

Second-Order Necessary Conditions for Optimal Control of Semilinear Elliptic Equations with Leading Term Containing Controls ^{*}

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Abstract. An optimal control problem for a semilinear elliptic equation of divergence form is considered. Both the leading term and the semilinear term of the state equation contain the control. The well-known Pontryagin type maximum principle for the optimal controls is the *first-order* necessary condition. When such a first-order necessary condition is singular in some sense, certain type of the *second-order* necessary condition will come in naturally. The aim of this paper is to explore such kind of conditions for our optimal control problem.

Keywords. optimal control, semilinear elliptic equation, control in leading term, second-order necessary conditions.

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1 Introduction.

Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with a smooth boundary $\partial\Omega$. Consider the following controlled elliptic partial differential equation (PDE, for short):

$$(1.1) \quad \begin{cases} -\nabla \cdot (A(x, u(x)) \nabla y(x)) = f(x, y(x), u(x)), & \text{in } \Omega, \\ y(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $A : \Omega \times U \rightarrow \mathbb{S}_+^n$ and $f : \Omega \times \mathbb{R} \times U \rightarrow \mathbb{R}$, with \mathbb{S}_+^n being the set of all $(n \times n)$ positive definite matrices, U being a separable (nonempty) metric space. In the above, $u(\cdot)$ is the *control* which belongs to the set \mathcal{U} of all *admissible controls* defined by the following:

$$\mathcal{U} \equiv \{u : \Omega \rightarrow U \mid u(\cdot) \text{ is measurable}\}.$$

Under some mild conditions, for any $u(\cdot) \in \mathcal{U}$, (1.1) admits a unique weak solution $y(\cdot) \equiv y(\cdot; u(\cdot))$ which is called the *state* (corresponding to the control $u(\cdot)$). The performance of the control $u(\cdot)$ is measured by the following cost functional

$$(1.2) \quad J(u(\cdot)) = \int_{\Omega} f^0(x, y(x), u(x)) dx \equiv \int_{\Omega} f^0(x, y(x; u(\cdot)), u(x)) dx$$

for some given map $f^0 : \Omega \times \mathbb{R} \times U \rightarrow \mathbb{R}$. Our optimal control problem can be stated as follows.

Problem (C). Find a $\bar{u}(\cdot) \in \mathcal{U}$ such that

$$(1.3) \quad J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}} J(u(\cdot)).$$

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Any $\bar{u}(\cdot) \in \mathcal{U}$ satisfying (1.3) is called an *optimal control*, and the corresponding $\bar{y}(\cdot) \equiv y(\cdot; \bar{u}(\cdot))$ is called an *optimal state*. The pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ is called an *optimal pair*.

Let us make some rough observations. Suppose $(\bar{y}(\cdot), \bar{u}(\cdot))$ is an optimal pair of Problem (C). For any given $u(\cdot) \in \mathcal{U}$, let $u^\delta(\cdot) \in \mathcal{U}$ be a suitable perturbation of $\bar{u}(\cdot)$ determined by $u(\cdot)$ with a parameter $\delta > 0$ (for examples, a convex type perturbation, or a spike type variation), so that $\tilde{\rho}(u^\delta(\cdot), \bar{u}(\cdot)) = O(\delta)$ with $\tilde{\rho}(\cdot, \cdot)$ being a suitable metric on the set \mathcal{U} , and the following holds:

$$(1.4) \quad J(u^\delta(\cdot)) = J(\bar{u}(\cdot)) + \delta J^1(\bar{u}(\cdot), u(\cdot)) + o(\delta), \quad \text{as } \delta \rightarrow 0.$$

Here, $J^1(u(\cdot), u(\cdot))$ is some functional of $(\bar{u}(\cdot), u(\cdot))$. The above can be called the first-order Taylor expansion of $J(\cdot)$ at $\bar{u}(\cdot)$, and $J^1(\bar{u}(\cdot), u(\cdot))$ can be regarded as the “directional derivative” of $J(\cdot)$ at $\bar{u}(\cdot)$ in the “direction” $u(\cdot)$. Hence, the minimality of $\bar{u}(\cdot)$ implies

$$(1.5) \quad J^1(\bar{u}(\cdot), u(\cdot)) \geq 0, \quad \forall u(\cdot) \in \mathcal{U}.$$

This is called the *first-order necessary condition* for $\bar{u}(\cdot)$, which is essentially the Pontryagin’s maximum principle for our Problem (C). Now, suppose that there is a set $\mathcal{U}_0 \subseteq \mathcal{U}$, which is different from the singleton $\{\bar{u}(\cdot)\}$, such that the following holds:

$$(1.6) \quad J^1(\bar{u}(\cdot), u(\cdot)) = 0, \quad \forall u(\cdot) \in \mathcal{U}_0.$$

Then $\bar{u}(\cdot)$ is said to be *singular* on the set \mathcal{U}_0 . For convenience, we call \mathcal{U}_0 a *singular set* of $\bar{u}(\cdot)$. Let

$$\mathcal{U}_0(\bar{u}(\cdot)) = \left\{ u(\cdot) \in \mathcal{U} \mid J^1(\bar{u}(\cdot), u(\cdot)) = 0 \right\},$$

which is called the *maximum singular set* of $\bar{u}(\cdot)$. When $\mathcal{U}_0(\bar{u}(\cdot)) = \mathcal{U}$, we say that $\bar{u}(\cdot)$ is *fully singular* (or simply *singular*); When $\mathcal{U}_0(\bar{u}(\cdot)) = \{\bar{u}(\cdot)\}$, we say that $\bar{u}(\cdot)$ is *nonsingular*; And, more interestingly, when $\{\bar{u}(\cdot)\} \neq \mathcal{U}_0(\bar{u}(\cdot)) \neq \mathcal{U}$, we say that $\bar{u}(\cdot)$ is *partially singular*. The notion of singular control was introduced by Gabasov–Kirillova in [17], where our partial singularity was called “the singularity in the sense of Pontryagin’s maximum principle”, and our full singularity was called “the singularity in the classical sense”. We prefer to use our shorter names. Now, suppose $\bar{u}(\cdot)$ is partially singular. Then one should expect that the following (comparing with (1.4))

$$(1.7) \quad J(u^\delta(\cdot)) = J(\bar{u}(\cdot)) + \delta^2 J^2(\bar{u}(\cdot), u(\cdot)) + o(\delta^2), \quad \forall u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot)),$$

for some functional $J^2(\bar{u}(\cdot), u(\cdot))$ of $(\bar{u}(\cdot), u(\cdot))$. The above can be called the second-order Taylor expansion of $J(\cdot)$ at $\bar{u}(\cdot)$ in the direction of $u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot))$, and $J^2(\bar{u}(\cdot), u(\cdot))$ can be regarded as the “second order directional derivative” at $\bar{u}(\cdot)$ in the “direction” of $u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot))$. Then the minimality of $\bar{u}(\cdot)$ leads to the following:

$$(1.8) \quad J^2(\bar{u}(\cdot), u(\cdot)) \geq 0, \quad \forall u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot)).$$

This is referred to as the *second-order necessary condition* of $\bar{u}(\cdot)$. We emphasize that the above holds only for all $u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot))$, the maximum singular set of $\bar{u}(\cdot)$. To get some more feeling, let us look at the following simple example, consisting of three situations.

Example 1.1. Let $U = [-1, 1] \times [-1, 1]$.

(i) Let $J(u) = u^2 = u_1^2 + u_2^2$ with $u = (u_1, u_2) \in U$. Then $u \mapsto J(u)$ is differentiable and the minimum is attained at $\bar{u} = (0, 0)$, an interior point of U , with $J(\bar{u}) = 0$. Therefore, for any $u = (u_1, u_2) \in U$ and $\delta \in (0, 1)$, we have $u^\delta = \bar{u} + \delta(u - \bar{u}) = \delta u \in U$, and

$$J(u^\delta) = \delta^2 u^2 \equiv J(\bar{u}) + \delta J^1(\bar{u}, u) + \delta^2 J^2(\bar{u}, u).$$

Consequently,

$$J^1(\bar{u}, u) = 0, \quad \forall u \in U.$$

This means that the maximum singular set $U_0(\bar{u})$ of \bar{u} coincides with U , and \bar{u} is fully singular. Hence,

$$J^2(\bar{u}, u) = u^2 \geq 0, \quad \forall u \in U_0(\bar{u}) = U,$$

which is the classical second-order necessary condition for \bar{u} .

(ii) Let $J(u) = u_1^3 + u_2^3$ with $u = (u_1, u_2) \in U$. The minimum is attained at $\bar{u} = (-1, -1)$. Then, for any $u = (u_1, u_2) \in U$ and any $\delta \in (0, 1) > 0$, one has

$$u^\delta = \bar{u} + \delta(u - \bar{u}) = (-1, -1) + \delta(u_1 + 1, u_2 + 1) \in U,$$

and

$$\begin{aligned} J(u^\delta) &= [-1 + \delta(u_1 + 1)]^3 + [-1 + \delta(u_2 + 1)]^3 \\ &= -2 + 3\delta(u_1 + u_2 + 2) - 3\delta^2[(u_1 + 1)^2 + (u_2 + 1)^2] + \delta^3[(u_1 + 1)^3 + (u_2 + 1)^3] \\ &= J(\bar{u}) + \delta J^1(\bar{u}, u) + \delta^2 J^2(\bar{u}, u) + \delta^3 J^3(\bar{u}, u). \end{aligned}$$

Thus,

$$J^1(\bar{u}, u) = u_1 + u_2 + 2 \neq 0, \quad \forall u = (u_1, u_2) \in U \setminus \{\bar{u}\}.$$

This means that \bar{u} is nonsingular. In this case, there is no second-order necessary condition for \bar{u} .

(iii) Let $J(u) = u_1^3 + u_2^2$ with $u = (u_1, u_2) \in U$. The minimum is attained at $\bar{u} = (-1, 0)$. For any $u = (u_1, u_2) \in U$, let the perturbation u^δ be defined by the following:

$$u^\delta = \bar{u} + \delta(u - \bar{u}) = (-1, 0) + \delta(u_1 + 1, u_2) = (-1 + \delta(u_1 + 1), \delta u_2) \in U.$$

Then we have

$$\begin{aligned} (1.9) \quad J(u^\delta) &= [-1 + \delta(u_1 + 1)]^3 + \delta^2 u_2^2 = J(\bar{u}) + 3\delta(u_1 + 1) + \delta^2[-3(u_1 + 1)^2 + u_2^2] + \delta^3(u_1 + 1)^3 \\ &\equiv J(\bar{u}) + \delta J^1(\bar{u}, u) + \delta^2 J^2(\bar{u}, u) + \delta^3 J^3(\bar{u}, u). \end{aligned}$$

Hence, the first-order necessary condition is

$$J^1(\bar{u}, u) \equiv 3(u_1 + 1) \geq 0, \quad \forall u = (u_1, u_2) \in U,$$

and \bar{u} is partially singular with $U_0(\bar{u}) = \{(-1, u_2) \mid u_2 \in [-1, 1]\}$. The second-order necessary condition is

$$J^2(\bar{u}, u) \equiv u_2^2 \geq 0, \quad \forall u \in U_0(\bar{u}).$$

However, we do not have (see (1.9))

$$J^2(\bar{u}, u) \equiv -3(u_1 + 1)^2 + u_2^2 \geq 0, \quad \forall u \in U.$$

The above example shows that in general, fully singular, partially singular, and nonsingular all can happen for a minimum of a function. Of course, the above is for scalar functions, and it is expected that the case of optimal control problems should be much more complicated. In Lou [24], second-order necessary and sufficient conditions for partially singular optimal controls of ordinary differential equations were established with general control domain. On the other hand, for convex control domains (mainly interval type), and mainly for fully singular cases, second-order necessary/sufficient optimality conditions have been studied for PDEs by many authors. We mention just a few of them here: Casas–Tröltzsch [9, 10, 11], Casas–Tröltzsch–Unger [14], Raymond–Tröltzsch [28], Mittelmann [27], Casas–Mateos [8], Rösch–Tröltzsch [29, 30], Wang–He [32], Casas–Los Reye–Tröltzsch [7], and Bonnans–Hermant [3, 4]. For some earlier works on ODEs, see Kelly [18], Kopp–Moyer [20], Gabasov–Kirillova [17], Krener [21], and Knobloch [19].

For problems of elliptic PDEs with control appearing in the leading term, Casas [5] studied the first-order necessary conditions for the case $A(x, u) = uI$ with quadratic cost functional and with the control being Lipschitz continuous. General case were treated by Lou–Yong in [26], and analogous results for parabolic and hyperbolic cases were given by Lou in [25] and Li-Lou in [22]. If the leading term of the state equation (1.1) does not contain controls, i.e., $A(x, u) \equiv A(x)$, then one can establish the second-order necessary conditions for partially singular optimal controls following similar arguments of [24]. However, if the leading term of the equation contains the control, we will see that it is much more complicated, even in defining the partial singularity of the optimal control. It turns out that the construction of a proper family of perturbations is much more difficult than the case without having control in the leading term, in order to have the first-order term disappeared in the Taylor type expansion. The difficult will be overcome by introducing the notion of weak singularity which involves a proper vector field. Consequently, the results obtained will have some big difference comparing with those for the problems without having the control in the leading term.

The rest of the paper is organized as follows. In Section 2, we will introduce the notions of singularity and weak singularity of the optimal controls. The main result of the paper will be stated, together with a couple of corollaries. Section 3 will be devoted to a review of the proof for the first-order necessary condition for Problem (C), which will inspire the second-order necessary condition. Section 4 is devoted to a proof of a result crucial for the proof of our main result. A proof of the second-order necessary condition will be presented in Section 5.

2 The Main Result.

For any differentiable function $\varphi : \Omega \rightarrow \mathbb{R}$, its gradient is denoted by $\nabla\varphi = (\frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_n})^\top : \Omega \rightarrow \mathbb{R}^n (\equiv \mathbb{R}^{n \times 1})$; For any differentiable vector-valued function $f = (f^1, f^2, \dots, f^n)^\top : \Omega \rightarrow \mathbb{R}^n$, its Jacobean matrix f_x is denoted by

$$f_x = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^2}{\partial x_1} & \cdots & \frac{\partial f^n}{\partial x_1} \\ \frac{\partial f^1}{\partial x_2} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^1}{\partial x_n} & \frac{\partial f^2}{\partial x_n} & \cdots & \frac{\partial f^n}{\partial x_n} \end{pmatrix} \equiv (\nabla f^1, \nabla f^2, \dots, \nabla f^n) \equiv \nabla f^\top.$$

Compatible with the above notation, we also will use

$$\nabla^\top F = (\nabla \cdot F^1, \nabla \cdot F^2, \dots, \nabla \cdot F^n).$$

for $F = (F^1, F^2, \dots, F^n) : \Omega \rightarrow \mathbb{R}^{n \times n}$. A function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be $[0, 1]^n$ -periodic if it admits a period 1 in every coordinate direction x_i , $i = 1, 2, \dots, n$. Denote $W_\#^{1,2}([0, 1]^n; \mathbb{R}^n)$ the space of all $[0, 1]^n$ -periodic vector-valued functions in $W_{loc}^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$ and $W_\#^{1,2}([0, 1]^n; \mathbb{R}^n)/\mathbb{R}^n$ the corresponding quotient space.

Next, let us introduce the following assumptions.

(S1) Set Ω is a bounded domain in \mathbb{R}^n ($n \geq 2$) with a smooth boundary $\partial\Omega$, and metric space (U, ρ) is separable.

(S2) Function $A : \Omega \times U \rightarrow \mathbb{S}_+^n$, (recall that \mathbb{S}_+^n is the set of all $(n \times n)$ (symmetric) positive definite matrices), for which $x \mapsto A(x, v)$ is measurable, and $v \mapsto A(x, v)$ is continuous. Further, there exist constants $\Lambda \geq \lambda > 0$ such that

$$(2.1) \quad \lambda|\xi|^2 \leq \langle A(x, v)\xi, \xi \rangle \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, v \in U.$$

(S3) Function $f : \Omega \times \mathbb{R} \times U \rightarrow \mathbb{R}$ has the following properties: $x \mapsto f(x, y, v)$ is measurable, $(y, v) \mapsto f(x, y, v)$ is continuous for almost all $x \in \Omega$, and $y \mapsto f(x, y, v)$ continuously differentiable. Moreover,

$$(2.2) \quad f_y(x, y, v) \leq 0, \quad \text{a.e. } (x, y, v) \in \Omega \times \mathbb{R} \times U$$

and for any $R > 0$, there exists a $M_R > 0$ such that

$$(2.3) \quad |f(x, y, v)| + |f_y(x, y, v)| \leq M_R, \quad \text{a.e. } (x, v) \in \Omega \times U, \quad |y| \leq R.$$

(S4) Function $f^0 : \Omega \times \mathbb{R} \times U \rightarrow \mathbb{R}$ has the following properties: $x \mapsto f^0(x, y, v)$ is measurable, $(y, v) \mapsto f^0(x, y, v)$ is continuous for almost all $x \in \Omega$, and $y \mapsto f^0(x, y, v)$ is continuously differentiable. Moreover, for any $R > 0$, there exists a $K_R > 0$ such that

$$(2.4) \quad |f^0(x, y, v)| + |f_y^0(x, y, v)| \leq K_R, \quad \text{a.e. } (x, v) \in \Omega \times U, \quad |y| \leq R.$$

It is standard that under (S1)–(S3), for any $u(\cdot) \in \mathcal{U}$, state equation (1.1) admits a unique weak solution $y(\cdot) = y(\cdot; u(\cdot)) \in H_0^1(\Omega) \cap C(\bar{\Omega})$ and the following estimate holds:

$$(2.5) \quad \|y(\cdot)\|_{H_0^1(\Omega)} + \|y(\cdot)\|_{L^\infty(\Omega)} \leq K,$$

for some constant $K > 0$. Therefore, if, in addition, (S4) is also assumed, then the cost functional is well-defined. Consequently, Problem (C) is well-formulated. The following was established in [26].

Theorem 2.1. *Let (S1)–(S4) hold. Let $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C), and $\bar{\psi}(\cdot)$ be the weak solution of the following adjoint equation:*

$$(2.6) \quad \begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla \bar{\psi}(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) \bar{\psi}(x) - f_y^0(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ \bar{\psi}|_{\partial\Omega} = 0. \end{cases}$$

Then

$$(2.7) \quad \begin{aligned} & H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), v) \\ & \geq \max_{\mu \in S^{n-1}} \frac{\langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{y}(x), \mu \rangle \langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{\psi}(x), \mu \rangle}{\langle A(x, v) \mu, \mu \rangle} \\ & = \frac{1}{2} |A(x, v)^{-\frac{1}{2}} (A(x, \bar{u}(x)) - A(x, v)) \nabla \bar{y}(x)| |A(x, v)^{-\frac{1}{2}} (A(x, \bar{u}(x)) - A(x, v)) \nabla \bar{\psi}(x)| \\ & \quad + \frac{1}{2} \langle A(x, v)^{-\frac{1}{2}} (A(x, \bar{u}(x)) - A(x, v)) \nabla \bar{y}(x), A(x, v)^{-\frac{1}{2}} (A(x, \bar{u}(x)) - A(x, v)) \nabla \bar{\psi}(x) \rangle \geq 0, \\ & \quad \forall v \in U, \quad \text{a.e. } x \in \Omega, \end{aligned}$$

where

$$(2.8) \quad \begin{aligned} H(x, y, \psi, \xi, \eta, v) &= \psi f(x, y, v) - f^0(x, y, v) - \langle A(x, v) \xi, \eta \rangle, \\ \forall (x, y, \psi, \xi, \eta, v) &\in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times U, \end{aligned}$$

which is called the Hamiltonian, and S^{n-1} is the unit sphere in \mathbb{R}^n .

The equality in (2.7) follows from the following simple fact (see [26], Lemma 2.3, for a proof).

$$(2.9) \quad \max_{\mu \in S^{n-1}} \langle \mu, \xi \rangle \langle \mu, \eta \rangle = \frac{|\xi| |\eta| + \langle \xi, \eta \rangle}{2}, \quad \forall \xi, \eta \in \mathbb{R}^n, \quad n \geq 2,$$

and the last inequality in (2.7) is due to the Cauchy-Schwartz inequality. Therefore, (2.7) implies (and might be a little stronger than) the following:

$$(2.10) \quad H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) = \max_{v \in U} H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), v), \quad \text{a.e. } x \in \Omega.$$

Let

$$\mathcal{L} = \left\{ \ell : \Omega \rightarrow S^{n-1} \mid \ell(\cdot) \text{ is measurable} \right\}.$$

We now introduce the following definition.

Definition 2.2. Let $\bar{u}(\cdot) \in \mathcal{U}$ be an optimal control of Problem (C).

(i) Let $(u(\cdot), \ell(\cdot)) \in \mathcal{U} \times \mathcal{L}$ satisfy the following:

$$(2.11) \quad \begin{aligned} & H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \\ &= \frac{\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \ell(x) \rangle \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \ell(x) \rangle}{\langle A(x, u(x)) \ell(x), \ell(x) \rangle}, \quad \text{a.e. } x \in \Omega. \end{aligned}$$

Then we say that $\bar{u}(\cdot)$ is *weakly singular* at $(u(\cdot), \ell(\cdot))$.

(ii) Let $u(\cdot) \in \mathcal{U}$ such that

$$(2.12) \quad H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) = H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)), \quad \text{a.e. } x \in \Omega,$$

then we say that $\bar{u}(\cdot)$ is *singular* at $u(\cdot)$.

(iii) Denote

$$\begin{aligned} \mathcal{V}_0(\bar{u}(\cdot)) &= \left\{ (u(\cdot), \ell(\cdot)) \in \mathcal{U} \times \mathcal{L} \mid \bar{u}(\cdot) \text{ is weakly singular at } (u(\cdot), \ell(\cdot)) \right\}, \\ \mathcal{U}_0(\bar{u}(\cdot)) &= \left\{ u(\cdot) \in \mathcal{U} \mid \bar{u}(\cdot) \text{ is singular at } u(\cdot) \right\}. \end{aligned}$$

If $\mathcal{V}_0(\bar{u}(\cdot)) = \mathcal{U} \times \mathcal{L}$, we say that $\bar{u}(\cdot)$ is *fully weakly singular*; If

$$\mathcal{V}_0(\bar{u}(\cdot)) \neq \mathcal{U} \times \mathcal{L}, \quad \mathcal{V}_0(\bar{u}(\cdot)) \setminus (\{\bar{u}(\cdot)\} \times \mathcal{L}) \neq \emptyset,$$

then we say that $\bar{u}(\cdot)$ is *partially weakly singular*; If $\mathcal{V}_0(\bar{u}(\cdot)) \subseteq \{\bar{u}(\cdot)\} \times \mathcal{L}$ (in this case, the equality actually holds), then we say that $\bar{u}(\cdot)$ is *weakly nonsingular*.

Likewise, we may define $\bar{u}(\cdot)$ to be *fully singular*, *partially singular*, and *nonsingular*, respectively, when $\mathcal{U}_0(\bar{u}(\cdot)) = \mathcal{U}$, $\{\bar{u}(\cdot)\} \neq \mathcal{U}_0(\bar{u}(\cdot)) \neq \mathcal{U}$ and $\mathcal{U}_0(\bar{u}(\cdot)) = \{\bar{u}(\cdot)\}$, respectively.

Let us make some observations on the above notions.

- If optimal control $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$ (with $u(\cdot) \neq \bar{u}(\cdot)$), then comparing (2.11) with (2.7), we see that for almost all $x \in \Omega$, $(u(x), \ell(x))$ is a maximum of the map

$$(v, \mu) \mapsto \frac{\langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{y}(x), \mu \rangle \langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{\psi}(x), \mu \rangle}{\langle A(x, v) \mu, \mu \rangle}$$

over $U \times S^{n-1}$. Note that such a maximum point might not be unique, in general.

- If optimal control $\bar{u}(\cdot)$ is singular at $u(\cdot) \neq \bar{u}(\cdot)$, then by (2.7), we see that

$$(2.13) \quad \begin{aligned} & \max_{\mu \in S^{n-1}} \frac{\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \mu \rangle \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \mu \rangle}{\langle A(x, u(x)) \mu, \mu \rangle} \\ &= \frac{1}{2} |A(x, u(x))^{-\frac{1}{2}} [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x)| |A(x, u(x))^{-\frac{1}{2}} [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x)| \\ & \quad + \frac{1}{2} \langle A(x, u(x))^{-\frac{1}{2}} [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), A(x, u(x))^{-\frac{1}{2}} [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x) \rangle = 0, \\ & \quad \text{a.e. } x \in \Omega. \end{aligned}$$

Therefore, by the compactness of S^{n-1} , together with Filipov's measurable selection lemma ([23]), we have some $\ell(\cdot) \in \mathcal{L}$ such that

$$(2.14) \quad \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \ell(x) \rangle \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \ell(x) \rangle = 0, \quad \text{a.e. } x \in \Omega.$$

Hence, the optimal control $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$ for some $\ell(\cdot) \in \mathcal{L}$.

• If optimal control $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$ such that (2.14) holds, then $\bar{u}(\cdot)$ must be singular at $u(\cdot)$. Further, it follows from the last equality in (2.13) that the equality holds in Cauchy-Schwartz inequality. Therefore,

$$[A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x) \quad \text{and} \quad [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x)$$

must be linearly dependent and have opposite directions. Consequently, (2.14) implies that

$$(2.15) \quad \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \ell(x) \rangle = \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \ell(x) \rangle = 0, \quad \text{a.e. } x \in \Omega.$$

The above can be summarized as follows.

Proposition 2.3. *Let (S1)–(S4) hold. Suppose $\bar{u}(\cdot)$ is an optimal control of Problem (C). If $\bar{u}(\cdot)$ is singular at $u(\cdot) \in \mathcal{U} \setminus \{\bar{u}(\cdot)\}$. Then there exists an $\ell(\cdot) \in \mathcal{L}$ such that (2.15) holds and $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$. Conversely, if $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot)) \in \mathcal{U} \times \mathcal{L}$ such that (2.15) holds, then $\bar{u}(\cdot)$ is singular at $u(\cdot)$.*

If $A(x, v)$ is independent of $v \in U$, then the right hand side of (2.11) is automatically zero, and (2.12) is true. Thus, in such a case, weak singularity is equivalent to singularity, and

$$\mathcal{V}_0(\bar{u}(\cdot)) = \mathcal{U}_0(\bar{u}(\cdot)) \times \mathcal{L}.$$

To state our main result of the current paper, the second-order necessary condition for optimal control of Problem (C), we need the following further assumption.

(S5) Function $y \mapsto (f(x, y, v), f^0(x, y, v))$ is twice continuously differentiable. Moreover, for any $R > 0$, there exists a $K_R > 0$ such that

$$(2.16) \quad |f_{yy}(x, y, v)| + |f_{yy}^0(x, y, u)| \leq K_R, \quad \forall v \in U, \quad |y| \leq R, \quad \text{a.e. } x \in \Omega.$$

We point out that, unlike most of the literature on PDE controls that we cited, no differentiability condition is assumed for the map $u \mapsto (f(x, y, u), f^0(x, y, u))$. Actually, our U is just a metric space which does not have a linear structure, in general. In particular, no convexity condition is assumed for U . Now, we state our main result of this paper.

Theorem 2.4. *Let (S1)–(S5) hold and $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C). Let $\bar{u}(\cdot)$ be partially weakly singular and $(u(\cdot), \ell(\cdot)) \in \mathcal{V}_0(\bar{u}(\cdot))$ with $u(\cdot) \neq \bar{u}(\cdot)$. Then the following holds:*

$$(2.17) \quad \begin{aligned} & \int_{\Omega} \left\{ \left(H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) \frac{\ell(x)^\top A(x, \bar{u}(x)) \ell(x)}{\ell(x)^\top A(x, u(x)) \ell(x)} \right. \\ & + \left(H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) Y(x) \\ & - \frac{1}{2} H_{yy}(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) |Y(x)|^2 \\ & \left. + \langle (A(x, u(x)) - A(x, \bar{u}(x))) \nabla \bar{\psi}(x), \nabla Y(x) \rangle \right\} dx \geq 0, \end{aligned}$$

where $\bar{\psi}(\cdot)$ is the weak solution to the adjoint equation (2.6), $H(\cdot)$ is the Hamiltonian defined by (2.8), and $Y(\cdot)$ is the weak solution to the following variational equation:

$$(2.18) \quad \begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla Y(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) Y(x) + \nabla \cdot (\Theta(x) \nabla \bar{y}(x)) \\ \quad + f(x, \bar{y}(x), u(x)) - f(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ Y|_{\partial\Omega} = 0, \end{cases}$$

with

$$(2.19) \quad \Theta(x) = A(x, u(x)) - A(x, \bar{u}(x)) - \frac{[A(x, u(x)) - A(x, \bar{u}(x))] \ell(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{\ell(x)^\top A(x, u(x)) \ell(x)}.$$

The proof of the above theorem will be carried out in Section 5. The following is a result concerning the partially singular (instead of partially weakly singular) optimal controls.

Corollary 2.5. *Let (S1)–(S5) hold, and $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C), with $\bar{u}(\cdot)$ being partially singular at $u(\cdot)$. Then*

$$(2.20) \quad \int_{\Omega} \left[\left(H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) Y(x) \right. \\ \left. - \frac{1}{2} H_{yy}(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) |Y(x)|^2 \right. \\ \left. + \langle (A(x, u(x)) - A(x, \bar{u}(x))) \nabla \bar{\psi}(x), \nabla Y(x) \rangle \right] dx \geq 0,$$

where $\bar{\psi}(\cdot)$ and $H(\cdot)$ are the same as those in Theorem 2.4, and $Y(\cdot)$ is the weak solution to the following variational equation:

$$(2.21) \quad \begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla Y(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) Y(x) + \nabla \cdot (A(x, u(x)) \nabla \bar{y}(x)) + f(x, \bar{y}(x), u(x)), & \text{in } \Omega, \\ Y|_{\partial\Omega} = 0. \end{cases}$$

Proof. By Proposition 2.3, we know that since $\bar{u}(\cdot)$ is singular at $u(\cdot)$, (2.15) holds for some $\ell(\cdot) \in \mathcal{L}$. Then

$$\nabla \cdot (\Theta(s) \nabla \bar{y}(x)) = \nabla \cdot ([A(x, u(x)) - A(x, \bar{u}(x))] \nabla \bar{y}(x)) = \nabla \cdot (A(x, u(x)) \nabla \bar{y}(x)) + f(x, \bar{y}(x), \bar{u}(x)).$$

Hence, (2.18) becomes (2.21), and (2.17) becomes (2.20) (making use the singularity of $\bar{u}(\cdot)$ at $u(\cdot)$, see (2.12)). Therefore, our conclusion follows. \square

The following gives the situation that the leading term does not contain the control, whose proof is pretty straightforward.

Corollary 2.6. *Let (S1)–(S5) hold with $A(x, v) \equiv A(x)$ independent of v . Let $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C), for which $\bar{u}(\cdot)$ is singular at $u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot)) \setminus \{\bar{u}(\cdot)\}$. Then*

$$(2.22) \quad \int_{\Omega} \left[\left(\mathcal{H}_y(x, \bar{y}(x), \bar{\psi}(x), \bar{u}(x)) - \mathcal{H}_y(x, \bar{y}(x), \bar{\psi}(x), u(x)) \right) Y(x) dx \right. \\ \left. - \frac{1}{2} \mathcal{H}_{yy}(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) |Y(x)|^2 \right] dx \geq 0,$$

where

$$(2.23) \quad \mathcal{H}(x, y, \psi, v) = \langle \psi, f(x, y, v) \rangle - f^0(x, y, v), \quad (x, y, \psi, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U,$$

and $\bar{\psi}(\cdot)$ and $Y(\cdot)$ are the weak solutions to the following adjoint equation and variational equation, respectively:

$$(2.24) \quad \begin{cases} -\nabla \cdot (A(x) \nabla \bar{\psi}(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) \bar{\psi}(x) - f_y^0(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ \bar{\psi}|_{\partial\Omega} = 0 \end{cases}$$

$$(2.25) \quad \begin{cases} -\nabla \cdot (A(x) \nabla Y(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) Y(x) + f(x, \bar{y}(x), u(x)) - f(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ Y|_{\partial\Omega} = 0. \end{cases}$$

3 The First-Order Necessary Condition Revisited

In this section, we briefly recall the proof of Theorem 2.1, from which we will find a correct direction approaching the second-order necessary condition for the optimal control. To this end, we first recall the following lemma ([1]).

Lemma 3.1. *Let $G : \Omega \times \mathbb{R}^n \rightarrow \mathbb{S}_+^n$ be measurable. Assume that*

(i) *$z \mapsto G(x, z)$ is $[0, 1]^n$ -periodic;*

(ii) *There exists two constants $\Lambda > \lambda > 0$, such that*

$$(3.1) \quad \lambda |\xi|^2 \leq \langle G(x, z) \xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall (x, z) \in \Omega \times \mathbb{R}^n, \xi \in \mathbb{R}^n;$$

(iii) *The following holds:*

$$(3.2) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \left| G\left(x, \frac{x}{\varepsilon}\right) \right|^2 dx = \int_{\Omega} \int_{[0,1]^n} |G(x, s)|^2 ds dx.$$

Let $g \in H^{-1}(\Omega)$ and $y_{\varepsilon}(\cdot)$ be the solution of

$$(3.3) \quad \begin{cases} -\nabla \cdot \left(G\left(x, \frac{x}{\varepsilon}\right) \nabla y_{\varepsilon}(x) \right) = g, & \text{in } \Omega, \\ y_{\varepsilon}|_{\partial\Omega} = 0. \end{cases}$$

Then $y_{\varepsilon}(\cdot)$ converges weakly to $y(\cdot)$ in $H_0^1(\Omega)$ where $y(\cdot)$ solves

$$(3.4) \quad \begin{cases} -\nabla \cdot (\widehat{G}(x) \nabla y(x)) = g, & \text{in } \Omega, \\ y|_{\partial\Omega} = 0, \end{cases}$$

and $\widehat{G}(x) \equiv (\widehat{G}_{ij}(x))$ is given by

$$(3.5) \quad \widehat{G}_{ij}(x) = \int_{[0,1]^n} \langle G(x, z) \nabla_z [\phi_i(x, z) + z_i], \nabla_z [\phi_j(x, z) + z_j] \rangle dz, \quad 1 \leq i, j \leq n,$$

with $\phi_i(x, \cdot) \in W_{\#}^{1,2}([0, 1]^n; \mathbb{R}^n)/\mathbb{R}^n$ being the unique solution of

$$(3.6) \quad -\nabla_z \cdot \left(G(x, z) \nabla_z [\phi_i(x, z) + z_i] \right) = 0, \quad 1 \leq i \leq n.$$

Observe that

$$\begin{aligned} & \int_{[0,1]^n} \langle G(x, z) \nabla_z [\phi_i(x, z) + z_i], \nabla_z [\phi_j(x, z) + z_j] \rangle dz \\ &= \int_{[0,1]^n} \left(\langle \nabla_z \phi_i(x, z), G(x, z) [\nabla_z \phi_j(x, z) + z_j] \rangle + \langle G(x, z) e_i, \nabla_z \phi_j(x, z) + e_j \rangle \right) dz \\ &= \int_{[0,1]^n} \left(\langle G(x, z) e_i, e_j \rangle + \langle G(x, z) e_i, \nabla_z \phi_j(x, z) \rangle \right) dz \\ &= \int_{[0,1]^n} \left(G_{ij}(x, z) + (G(x, z) \nabla_z \phi(x, z))^{\top}_{ij} \right) dz. \end{aligned}$$

Hence,

$$(3.7) \quad \widehat{G}(x) = \int_{[0,1]^n} G(x, z) \left(I + \nabla_z \phi(x, z)^{\top} \right) dz.$$

Also, (3.6) can be written as

$$(3.8) \quad -\nabla_z^\top \left[G(x, z)(I + \nabla_z \phi^\top) \right] = 0.$$

Note that in general $G(x, \frac{x}{\varepsilon})$ does not necessarily converge strongly in $L^2(\Omega)$ (as $\varepsilon \downarrow 0$). Therefore, the above lemma is by no means trivial or obvious. On the other hand, the following result is much easier, which will also be used later, for different situations.

Lemma 3.2. *Let $\varepsilon > 0$ and $G_\varepsilon(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n)$. Assume that there exist two constants $\Lambda > \lambda > 0$, such that*

$$\lambda|\xi|^2 \leq \langle G_\varepsilon(x)\xi, \xi \rangle \leq \Lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n;$$

and $G_\varepsilon(\cdot)$ converges to $G(\cdot)$ strongly in $L^2(\Omega)$. Let $g \in H^{-1}(\Omega)$ and $y_\varepsilon(\cdot)$ be the solution of

$$\begin{cases} -\nabla \cdot (G_\varepsilon(x) \nabla y_\varepsilon(x)) = g, & \text{in } \Omega, \\ y_\varepsilon|_{\partial\Omega} = 0. \end{cases}$$

Then $y_\varepsilon(\cdot)$ converges strongly to $y(\cdot)$ in $H_0^1(\Omega)$, as $\varepsilon \downarrow 0$, where $y(\cdot)$ solves

$$\begin{cases} -\nabla \cdot (G(x) \nabla y(x)) = g, & \text{in } \Omega, \\ y|_{\partial\Omega} = 0. \end{cases}$$

In the above lemma, one can prove easily that $y_\varepsilon(\cdot)$ converges weakly to $y(\cdot)$ in $H_0^1(\Omega)$, as $\varepsilon \downarrow 0$. The strong convergence follows from

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \int_{\Omega} \langle G_\varepsilon(x) (\nabla y_\varepsilon(x) - \nabla y(x)), \nabla y_\varepsilon(x) - \nabla y(x) \rangle dx \\ &= \lim_{\varepsilon \downarrow 0} \int_{\Omega} \langle G_\varepsilon(x) \nabla y_\varepsilon(x), \nabla y_\varepsilon(x) - \nabla y(x) \rangle dx = \lim_{\varepsilon \downarrow 0} \int_{\Omega} (y_\varepsilon(x) - y(x)) g dx = 0. \end{aligned}$$

We now recall the proof of Theorem 2.1 (see [26] for technical details). Let $\bar{u}(\cdot) \in \mathcal{U}$ be an optimal control and $u(\cdot) \in \mathcal{U}$ be an arbitrary fixed control. Pick any $\mu \in S^{n-1}$. Define a two-parameter spike variation $u^{\alpha, \varepsilon}(\cdot; \mu)$ of the control $\bar{u}(\cdot)$ associated with $u(\cdot)$ and μ as follows:

$$(3.9) \quad u^{\alpha, \varepsilon}(x; \mu) = \begin{cases} u(x), & \text{if } \left\{ \frac{\langle x, \mu \rangle}{\varepsilon} \right\} \in [0, \alpha), \\ \bar{u}(x), & \text{if } \left\{ \frac{\langle x, \mu \rangle}{\varepsilon} \right\} \in [\alpha, 1), \end{cases}$$

where $\{a\} \equiv a - [a]$ denotes the decimal part of the real number a . Then $u^{\alpha, \varepsilon}(\cdot; \mu) \in \mathcal{U}$. Here, the dependence on μ is emphasized. We should keep in mind that $u^{\alpha, \varepsilon}(\cdot; \mu)$ also depends on the selected control $u(\cdot)$ (which is fixed). Let $y^{\alpha, \varepsilon}(\cdot; \mu) = y(\cdot; u^{\alpha, \varepsilon}(\cdot; \mu))$ be the state corresponding to the control $u^{\alpha, \varepsilon}(\cdot; \mu)$. Then by Lemma 3.1, as $\varepsilon \downarrow 0$, $y^{\alpha, \varepsilon}(\cdot; \mu)$ converges to $y^\alpha(\cdot; \mu)$, weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, where $y^\alpha(\cdot; \mu)$ solves the following PDE, which is called a *relaxed state equation*:

$$(3.10) \quad \begin{cases} -\nabla \cdot (A^\alpha(x; \mu) \nabla y^\alpha(x; \mu)) = (1 - \alpha)f(x, y^\alpha(x; \mu), \bar{u}(x)) + \alpha f(x, y^\alpha(x; \mu), u(x)), & \text{in } \Omega, \\ y^\alpha(x; \mu) = 0, & \text{in } \partial\Omega, \end{cases}$$

with

$$(3.11) \quad \begin{aligned} A^\alpha(x; \mu) &= \alpha A(x, u(x)) + (1 - \alpha)A(x, \bar{u}(x)) \\ &\quad - \frac{\alpha(1 - \alpha)[A(x, u(x)) - A(x, \bar{u}(x))] \mu \mu^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{(1 - \alpha)\mu^\top A(x, u(x))\mu + \alpha\mu^\top A(x, \bar{u}(x))\mu}. \end{aligned}$$

Define

$$Y^\alpha(\cdot; \mu) = \frac{y^\alpha(\cdot; \mu) - \bar{y}(\cdot)}{\alpha}.$$

Then, as $\alpha \downarrow 0$, $Y^\alpha(\cdot; \mu)$ converges to $Y(\cdot; \mu)$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, where $Y(\cdot; \mu)$ is the weak solution to the following:

$$(3.12) \quad \begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla Y(x; \mu)) = f_y(x, \bar{y}(x), \bar{u}(x)) Y(x; \mu) + \nabla \cdot (\Theta(x; \mu) \nabla \bar{y}(x)) \\ \quad + f(x, \bar{y}(x), u(x)) - f(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ Y(x; \mu) = 0, & \text{in } \partial\Omega, \end{cases}$$

with

$$(3.13) \quad \Theta(x; \mu) = A(x, u(x)) - A(x, \bar{u}(x)) - \frac{[A(x, u(x)) - A(x, \bar{u}(x))] \mu \mu^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{\mu^\top A(x, u(x)) \mu}.$$

Consequently, as $\alpha \downarrow 0$, $y^\alpha(\cdot; \mu)$ converges to $\bar{y}(\cdot)$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. On the other hand, by the convergence of $y^{\alpha, \varepsilon}(\cdot; \mu) \rightarrow y^\alpha(\cdot; \mu)$ (strongly in $L^2(\Omega)$, as $\varepsilon \downarrow 0$), one has

$$(3.14) \quad \lim_{\varepsilon \downarrow 0} J(u^{\alpha, \varepsilon}(\cdot; \mu)) = J^\alpha(u(\cdot), \mu) \equiv \alpha \int_\Omega f^0(x, y^\alpha(x; \mu), u(x)) dx + (1 - \alpha) \int_\Omega f^0(x, y^\alpha(x; \mu), \bar{u}(x)) dx.$$

Further, (suppressing x)

$$(3.15) \quad \begin{aligned} J^\alpha(u(\cdot), \mu) - J(\bar{u}(\cdot)) &= \int_\Omega [\alpha (f^0(y^\alpha, u) - f^0(y^\alpha, \bar{u})) + f^0(y^\alpha, \bar{u}) - f^0(\bar{y}, \bar{u})] dx \\ &= \alpha \int_\Omega \left[(f^0(y^\alpha, u) - f^0(y^\alpha, \bar{u})) + \left(\int_0^1 f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) dt \right) Y^\alpha \right] dx. \end{aligned}$$

Thus,

$$(3.16) \quad \lim_{\alpha \downarrow 0} \frac{J^\alpha(u(\cdot), \mu) - J(\bar{u}(\cdot))}{\alpha} = \int_\Omega [f^0(x, \bar{y}(x), u(x)) - f^0(x, \bar{y}(x), \bar{u}(x)) + f_y^0(x, \bar{y}(x), \bar{u}(x)) Y(x; \mu)] dx.$$

Consequently,

$$(3.17) \quad \begin{aligned} J(u^{\alpha, \varepsilon}(\cdot; \mu)) - J(\bar{u}(\cdot)) &= J^\alpha(u(\cdot), \mu) - J(\bar{u}(\cdot)) + r_\varepsilon \\ &= \alpha \int_\Omega [f^0(x, \bar{y}(x), u(x)) - f^0(x, \bar{y}(x), \bar{u}(x)) + f_y^0(x, \bar{y}(x), \bar{u}(x)) Y(x; \mu)] dx + \alpha \rho_\alpha + r_\varepsilon, \end{aligned}$$

where

$$\lim_{\varepsilon \downarrow 0} r_\varepsilon = 0, \quad \lim_{\alpha \downarrow 0} \rho_\alpha = 0.$$

By the duality, we obtain

$$(3.18) \quad \begin{aligned} J(u^{\alpha, \varepsilon}(\cdot; \mu)) &= J(\bar{u}(\cdot)) \\ &\quad + \alpha \int_\Omega [H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x))] \\ &\quad - \frac{\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \mu \rangle \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \mu \rangle}{\langle A(x, u(x)) \mu, \mu \rangle} dx + \alpha \rho_\alpha + r_\varepsilon \\ &\equiv J(\bar{u}(\cdot)) + \alpha J^1(\bar{u}(\cdot); u(\cdot), \mu) + \alpha \rho_\alpha + r_\varepsilon. \end{aligned}$$

By the optimality of $\bar{u}(\cdot)$, the above leads to the following:

$$J^1(\bar{u}(\cdot); u(\cdot), \mu) \geq 0,$$

which implies

$$(3.19) \quad \begin{aligned} & H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), v) \\ & - \frac{\langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{y}(x), \mu \rangle \langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{\psi}(x), \mu \rangle}{\langle A(x, v) \mu, \mu \rangle} \geq 0, \\ & \forall v \in U, \quad \mu \in S^{n-1}, \quad \text{a.e. } x \in \Omega. \end{aligned}$$

Hence, (2.7) follows, proving the first-order necessary condition.

Note that in the above result, $\mu \in S^{n-1}$ is a given fixed direction. Whereas, when an optimal control $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot)) \in \mathcal{V}_0(\bar{u}(\cdot))$, $\ell(\cdot) \in \mathcal{L}$ might not be a fixed μ . Therefore, we need to extend (3.18), allowing μ to be replaced by $\ell(\cdot) \in \mathcal{L}$. More precisely, we hope to have the following result.

Proposition 3.3. *Let (S1)–(S4) hold and $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C). Let $(u(\cdot), \ell(\cdot)) \in \mathcal{U} \times \mathcal{L}$, and let $y^\alpha(\cdot; \ell(\cdot))$ be the weak solution to the following equation:*

$$(3.20) \quad \begin{cases} -\nabla \cdot (A^\alpha(x; \ell(\cdot)) \nabla y^\alpha(x; \ell(\cdot))) = (1 - \alpha)f(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) + \alpha f(x, y^\alpha(x; \ell(\cdot)), u(x)), & \text{in } \Omega, \\ y^\alpha(x; \ell(\cdot)) = 0, & \text{in } \partial\Omega. \end{cases}$$

with

$$(3.21) \quad \begin{aligned} A^\alpha(x; \ell(\cdot)) &= \alpha A(x, u(x)) + (1 - \alpha)A(x, \bar{u}(x)) \\ &- \frac{\alpha(1 - \alpha)[A(x, u(x)) - A(x, \bar{u}(x))] \ell(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{(1 - \alpha)\ell(x)^\top A(x, u(x))\ell(x) + \alpha\ell(x)^\top A(x, \bar{u}(x))\ell(x)}. \end{aligned}$$

Define

$$(3.22) \quad J^\alpha(u(\cdot); \ell(\cdot)) = \alpha \int_\Omega f^0(s, y^\alpha(x; \ell(\cdot)), u(x)) dx + (1 - \alpha) \int_\Omega f^0(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) dx.$$

Then

$$(3.23) \quad \begin{aligned} & J^\alpha(u(\cdot), \ell(\cdot)) = J(\bar{u}(\cdot)) \\ & + \alpha \int_\Omega \left[H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right. \\ & \left. - \frac{\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \ell(x) \rangle \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \ell(x) \rangle}{\langle A(x, u(x)) \ell(x), \ell(x) \rangle} \right] dx + \alpha \rho_\alpha, \end{aligned}$$

and

$$(3.24) \quad J^\alpha(u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot)) \geq 0.$$

If the above result holds true, then in the case that $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$, the above (3.23) will become the following in which the first order term disappears

$$J^\alpha(u(\cdot), \ell(\cdot)) = J(\bar{u}(\cdot)) + \alpha \rho_\alpha.$$

To further characterize the optimal control, the second-order necessary condition will be needed.

To prove Proposition 3.3, it is natural to try a modification of (3.9) as follows:

$$(3.25) \quad u^{\alpha, \varepsilon}(x) = \begin{cases} u(x), & \text{if } \left\{ \frac{\langle x, \ell(x) \rangle}{\varepsilon} \right\} \in [0, \alpha), \\ \bar{u}(x), & \text{if } \left\{ \frac{\langle x, \ell(x) \rangle}{\varepsilon} \right\} \in [\alpha, 1), \end{cases}$$

and wish that $y^{\alpha,\varepsilon}(\cdot) = y(\cdot; u^{\alpha,\varepsilon}(\cdot))$ converges to $y^\alpha(\cdot)$ weakly in $H_0^1(\Omega)$ (as $\varepsilon \downarrow 0$) with $y^\alpha(\cdot)$ being the weak solution to (3.20). However, (3.25) does not work as we expected. For example, for $n = 2$, let

$$\ell(x_1, x_2) = \frac{(-x_2, x_1)}{\sqrt{x_1^2 + x_2^2}}, \quad \forall x \neq 0.$$

Then (3.25) implies

$$u^{\alpha,\varepsilon}(x) = u(x), \quad \forall x \in \Omega.$$

Thus, in this case, $u^{\alpha,\varepsilon}(\cdot)$ is not a proper perturbation of $\bar{u}(\cdot)$ that we expected since for any proper metric $\tilde{\rho}$ on \mathcal{U} ,

$$\tilde{\rho}(u^{\varepsilon,\alpha}(\cdot), \bar{u}(\cdot)) = \tilde{\rho}(u(\cdot), \bar{u}(\cdot)),$$

which will not go to zero as $\alpha, \varepsilon \downarrow 0$. Nevertheless, in the next section, we will prove that Proposition 3.3 is true, by a different method.

4 Proof of Proposition 3.3.

In this section, we will present a proof of Proposition 3.3. Let us begin with the following lemma.

Lemma 4.1. *Let $B_1, B_2 \in \mathbb{S}_+^n$ and*

$$\lambda|\xi|^2 \leq \langle B_i \xi, \xi \rangle \leq \Lambda|\xi|^2, \quad \forall i = 1, 2, \quad \xi \in \mathbb{R}^n$$

for some constants $\Lambda \geq \lambda > 0$. Let $\alpha \in (0, 1)$, $\mu \in \mathbb{R}^n \setminus \{0\}$ and

$$G = \alpha B_1 + (1 - \alpha)B_2 - \alpha(1 - \alpha) \frac{(B_2 - B_1)\mu\mu^\top(B_2 - B_1)}{\mu^\top[\alpha B_2 + (1 - \alpha)B_1]\mu}.$$

Then

$$(4.1) \quad (\alpha B_1^{-1} + (1 - \alpha)B_2^{-1})^{-1} \leq G \leq \alpha B_1 + (1 - \alpha)B_2.$$

Proof. The lemma is a consequence of Theorem 1.3.14 and Lemma 1.3.32 in [2]. Here we give a direct proof of it. Let

$$C \equiv \alpha B_2 + (1 - \alpha)B_1 = B_1 + \alpha(B_2 - B_1) = B_2 - (1 - \alpha)(B_2 - B_1).$$

Then $C \in \mathbb{S}_+^n$, and

$$C - B_1 = \alpha(B_2 - B_1), \quad C - B_2 = -(1 - \alpha)(B_2 - B_1).$$

Thus,

$$(4.2) \quad \begin{aligned} & \alpha B_1 + (1 - \alpha)B_2 - \alpha(1 - \alpha)(B_2 - B_1)C^{-1}(B_2 - B_1) \\ &= \alpha B_1 + (1 - \alpha)B_2 + (C - B_1)C^{-1}(C - B_2) \\ &= \alpha B_1 + (1 - \alpha)B_2 + C - B_1 - B_2 + B_1C^{-1}B_2 \\ &= B_1[\alpha B_2 + (1 - \alpha)B_1]^{-1}B_2 = [\alpha B_1^{-1} + (1 - \alpha)B_2^{-1}]^{-1}. \end{aligned}$$

Consequently, we get (4.1) since

$$(4.3) \quad 0 \leq \frac{\mu\mu^\top}{\mu^\top[\alpha B_2 + (1 - \alpha)B_1]\mu} = \frac{\mu\mu^\top}{\mu^\top C \mu} = C^{-\frac{1}{2}} \frac{C^{\frac{1}{2}}\mu}{|C^{\frac{1}{2}}\mu|} \left(\frac{C^{\frac{1}{2}}\mu}{|C^{\frac{1}{2}}\mu|} \right)^\top C^{-\frac{1}{2}} \leq C^{-1},$$

proving our conclusion. \square

The following lemma will play an interesting role blow.

Lemma 4.2. Let $\nu = (\nu_1, \dots, \nu_n)^\top \in \mathbb{Z}^n \setminus \{0\}$ (where \mathbb{Z} is the set of all integers). Then for any $\alpha \in (0, 1)$,

$$\int_{[0,1]^n} \chi_{[0,\alpha)}(\{\langle \nu, z \rangle\}) dz = \alpha.$$

Proof. Recall that $\{a\} = a - [a]$ is the decimal part of the real number a . Note that

$$\langle \nu, z \rangle = \sum_{k=1}^n \nu_k z_k,$$

If some of integers ν_k are zero, we could drop the corresponding terms and reduce the dimension of z . Thus, we assume all ν_k are non-zero. Also, if some $\nu_k < 0$, we may replace corresponding z_k by $(1 - z_k)$. Therefore, we may let all $\nu_k > 0$. Next, we observe the following (noting the $[0, 1]^n$ -periodicity of the maps $z \mapsto \{\langle \nu, z \rangle\}$):

$$\begin{aligned} \int_{[0,1]^n} \chi_{[0,\alpha)}(\{\langle \nu, z \rangle\}) dz &= \int_0^1 dz_1 \int_0^1 dz_2 \cdots \int_0^1 \chi_{[0,\alpha)}\left(\left\{\sum_{k=1}^n \nu_k z_k\right\}\right) dz_n \\ &= \nu_1 \nu_2 \cdots \nu_n \int_0^{\frac{1}{\nu_1}} dz_1 \int_0^{\frac{1}{\nu_2}} dz_2 \cdots \int_0^{\frac{1}{\nu_n}} \chi_{[0,\alpha)}\left(\left\{\sum_{k=1}^n \nu_k z_k\right\}\right) dz_n \\ &= \int_0^1 dz_1 \int_0^1 dz_2 \cdots \int_0^1 \chi_{[0,\alpha)}\left(\left\{\sum_{k=1}^n z_k\right\}\right) dz_n. \end{aligned}$$

Hence, we need to prove the following:

$$\int_0^1 dz_1 \int_0^1 dz_2 \cdots \int_0^1 \chi_{[0,\alpha)}\left(\left\{\sum_{k=1}^n z_k\right\}\right) dz_n = \alpha.$$

Let us use induction. For $n = 1$, the above is clearly true. Suppose the above holds for $n - 1$. Then, for the n -dimensional case, we observe the following: For $z_1 \in [0, 1)$,

$$\left\{\sum_{k=1}^n z_k\right\} = \left\{z_1 + \sum_{k=2}^n z_k\right\} = \begin{cases} z_1 + \left\{\sum_{k=2}^n z_k\right\}, & \text{if } 0 \leq z_1 + \left\{\sum_{k=2}^n z_k\right\} < 1, \\ z_1 + \left\{\sum_{k=2}^n z_k\right\} - 1, & \text{if } 1 \leq z_1 + \left\{\sum_{k=2}^n z_k\right\} < 2. \end{cases}$$

Then for $z_1 \in [0, \alpha)$, the following holds

$$\chi_{[0,\alpha)}\left(\left\{z_1 + \sum_{k=2}^n z_k\right\}\right) = 1,$$

if and only if either

$$0 \leq z_1 + \left\{\sum_{k=2}^n z_k\right\} < \alpha \quad \text{and} \quad 0 \leq z_1 + \left\{\sum_{k=2}^n z_k\right\} < 1.$$

or

$$0 \leq z_1 + \left\{\sum_{k=2}^n z_k\right\} - 1 < \alpha \quad \text{and} \quad 1 \leq z_1 + \left\{\sum_{k=2}^n z_k\right\} < 2.$$

That is, either

$$0 \leq \left\{\sum_{k=2}^n z_k\right\} < \alpha - z_1,$$

or

$$1 - z_1 \leq \left\{ \sum_{k=2}^n z_k \right\} \leq (\alpha + 1 - z_1) \wedge 1 = 1.$$

Note that the above two cases are mutually exclusive (since $\alpha - z_1 < 1 - z_1$). On the other hand, for $z_1 \in [\alpha, 1)$, if $0 \leq z_1 + \left\{ \sum_{k=2}^n z_k \right\} < 1$, then

$$\left\{ z_1 + \sum_{k=2}^n z_k \right\} = z_1 + \left\{ \sum_{k=2}^n z_k \right\} \geq \alpha \Rightarrow \chi_{[0, \alpha)} \left(z_1 + \left\{ \sum_{k=2}^n z_k \right\} \right) = 0.$$

Thus the following holds:

$$\chi_{[0, \alpha)} \left(\left\{ z_1 + \sum_{k=2}^n z_k \right\} \right) = 1,$$

if and only if

$$0 \leq z_1 + \left\{ \sum_{k=2}^n z_k \right\} - 1 < \alpha, \quad 1 \leq z_1 + \left\{ \sum_{k=2}^n z_k \right\} < 2.$$

That is,

$$0 \leq 1 - z_1 \leq \left\{ \sum_{k=2}^n z_k \right\} \leq \alpha + 1 - z_1 \leq 1.$$

Hence, by induction hypothesis,

$$\begin{aligned} \int_{[0,1]^n} \chi_{[0, \alpha)} \left(\left\{ z_1 + \sum_{k=2}^n z_k \right\} \right) dz &= \int_0^\alpha dz_1 \int_{[0,1]^{n-1}} \chi_{[0, \alpha - z_1)} \left(\left\{ \sum_{k=2}^n z_k \right\} \right) dz_2 \cdots dz_n \\ &\quad + \int_0^\alpha dz_1 \int_{[0,1]^{n-1}} \chi_{[1 - z_1, 1)} \left(\left\{ \sum_{k=2}^n z_k \right\} \right) dz_2 \cdots dz_n \\ &\quad + \int_\alpha^1 dz_2 \int_{[0,1]^{n-1}} \chi_{[1 - z_1, \alpha + 1 - z_1)} \left(\left\{ \sum_{k=1}^n z_k \right\} \right) dz_2 \cdots dz_n \\ &= \int_0^\alpha (\alpha - z_1) dz_1 + \int_0^\alpha z_1 dz_1 + \int_\alpha^1 \alpha dz_1 = \alpha. \end{aligned}$$

This completes the proof. \square

The following gives a crucial convergence of the weak solution to the state equation under a suitable perturbation of the leading coefficient.

Lemma 4.3. *Let (S1) hold. Let $B(\cdot) = (b_{ij}(\cdot))$, $C(\cdot) = (c_{ij}(\cdot)) \in L^\infty(\Omega; \mathbb{S}_+^n)$ such that*

$$(4.4) \quad \begin{aligned} \lambda |\xi|^2 &\leq \langle B(x)\xi, \xi \rangle \leq \Lambda |\xi|^2, \\ \lambda |\xi|^2 &\leq \langle C(x)\xi, \xi \rangle \leq \Lambda |\xi|^2, \end{aligned} \quad \forall (\xi, x) \in \mathbb{R}^n \times \Omega,$$

for some $\Lambda \geq \lambda > 0$. Let $\alpha \in (0, 1)$, $\mu_k = \left(\frac{r_{k1}}{p_{k1}}, \frac{r_{k2}}{p_{k2}}, \dots, \frac{r_{kn}}{p_{kn}} \right) \in \mathbb{Q}^n$, \mathbb{Q} is the set of all rational numbers, with all r_{kj} being integers, and p_{kj} being positive integers, $1 \leq k \leq m$, $1 \leq j \leq n$. Let E_1, E_2, \dots, E_m be mutually disjoint measurable sets such that $\bigcup_{k=1}^m E_k = \Omega$. Let $h(\cdot) \in L^2(\Omega)$, and

$$(4.5) \quad G(x, z) = \begin{cases} B(x), & \text{if } \langle z, \mu_k \rangle \in [0, \alpha), \\ C(x), & \text{if } \langle z, \mu_k \rangle \in [\alpha, 1), \end{cases} \quad x \in E_k, \quad 1 \leq k \leq m.$$

For $\varepsilon > 0$, let $y^\varepsilon(\cdot) \in H_0^1(\Omega)$ be the weak solution of the following:

$$(4.6) \quad \begin{cases} -\nabla \cdot \left(G(x, \frac{x}{\varepsilon}) \nabla y^\varepsilon(x) \right) = h(x), & \text{in } \Omega, \\ y^\varepsilon|_{\partial\Omega} = 0. \end{cases}$$

Then, as $\varepsilon \downarrow 0$,

$$(4.7) \quad y^\varepsilon(\cdot) \rightarrow y(\cdot), \quad \text{weakly in } H_0^1(\Omega),$$

where $y(\cdot)$ is the weak solution of

$$(4.8) \quad \begin{cases} -\nabla \cdot (\widehat{G}(x) \nabla y(x)) = h(x), & \text{in } \Omega, \\ y|_{\partial\Omega} = 0 \end{cases}$$

with $\widehat{G}(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n)$ given by

$$(4.9) \quad \widehat{G}(x) = \alpha B(x) + (1 - \alpha)C(x) - \frac{\alpha(1 - \alpha)[B(x) - C(x)]\ell(x)\ell(x)^\top[B(x) - C(x)]}{(1 - \alpha)\ell(x)^\top B(x)\ell(x) + \alpha\ell(x)^\top C(x)\ell(x)}, \quad x \in \Omega,$$

and

$$\ell(x) = \sum_{k=1}^m \mu_k \chi_{E_k}(x), \quad x \in \Omega.$$

Proof. The proof is essentially inspired by that of Lemma 1.3.32 of [2]. Let P be a common multiple of p_{kj} , $k = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. Then one can verify that $G(x, Pz)$ is $[0, 1]^n$ -periodic in z , for any $\varepsilon > 0$, $G(x, \frac{Px}{\varepsilon})$ is measurable, and

$$\lambda|\xi|^2 \leq \langle G(x, Pz)\xi, \xi \rangle \leq \Lambda|\xi|^2, \quad \forall (x, z) \in \Omega \times \mathbb{R}^n, \quad \xi \in \mathbb{R}^n.$$

Moreover, using Riemann-Lebesgue's Theorem (see Ch. II, Theorem 4.15 in [33]),

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\Omega} \left| G\left(x, \frac{Px}{\varepsilon}\right) \right|^2 dx &= \lim_{\varepsilon \downarrow 0} \sum_{k=1}^m \int_{E_k} \left| G\left(x, \frac{Px}{\varepsilon}\right) \right|^2 dx \\ &= \lim_{\varepsilon \downarrow 0} \sum_{k=1}^m \int_{E_k} \left| B(x) \chi_{[0, \alpha)} \left(\left\{ \left\langle \frac{Px}{\varepsilon}, \mu_k \right\rangle \right\} \right) + C(x) \chi_{[\alpha, 1)} \left(\left\{ \left\langle \frac{Px}{\varepsilon}, \mu_k \right\rangle \right\} \right) \right|^2 dx \\ &= \lim_{\varepsilon \downarrow 0} \sum_{k=1}^m \int_{E_k} \left[|B(x)|^2 \chi_{[0, \alpha)} \left(\left\{ \left\langle \frac{Px}{\varepsilon}, \mu_k \right\rangle \right\} \right) + |C(x)|^2 \chi_{[\alpha, 1)} \left(\left\{ \left\langle \frac{Px}{\varepsilon}, \mu_k \right\rangle \right\} \right) \right] dx \\ &= \sum_{k=1}^m \left[\int_{E_k} |B(x)|^2 dx \int_{[0, 1]^n} \chi_{[0, \alpha)} \left(\left\{ \langle Pz, \mu_k \rangle \right\} \right) dz + \int_{E_k} |C(x)|^2 dx \int_{[0, 1]^n} \chi_{[\alpha, 1)} \left(\left\{ \langle Pz, \mu_k \rangle \right\} \right) dz \right] \\ &= \sum_{k=1}^m \int_{E_k} \left[\int_{[0, 1]^n} \left| B(x) \chi_{[0, \alpha)} \left(\left\{ \langle Pz, \mu_k \rangle \right\} \right) + C(x) \chi_{[\alpha, 1)} \left(\left\{ \langle Pz, \mu_k \rangle \right\} \right) \right|^2 dz \right] dx = \int_{\Omega} \int_{[0, 1]^n} |G(x, Pz)|^2 dz dx. \end{aligned}$$

Thus, $(x, z) \mapsto G(x, Pz)$ satisfies conditions of Lemma 3.1. Using Lemma 3.1, we get that (4.7)–(4.8) with

$$(4.10) \quad \widehat{G}(x) = \int_{[0, 1]^n} G(x, Pz) \left[I + \nabla_z \phi(x, z)^\top \right] dz,$$

with $\phi \in W_{\#}^{1,2}([0, 1]^n; \mathbb{R}^n)/\mathbb{R}^n$ being the unique solution of

$$(4.11) \quad -\nabla_z^\top \left[G(x, Pz) \left(I + \nabla_z \phi(x, z)^\top \right) \right] = 0.$$

That is, for $x \in E_k$,

$$(4.12) \quad \nabla_z^\top \left[\left(\chi_{[0,\alpha]}(\{\langle Pz, \mu_k \rangle\}) B(x) + \chi_{[\alpha,1]}(\{\langle Pz, \mu_k \rangle\}) C(x) \right) \left(I + \nabla_z \phi(x, z)^\top \right) \right] = 0.$$

To solve (4.12), we fix k , denote $\tilde{\mu} = \frac{\mu_k}{|\mu_k|}$ and define

$$\varphi(x, z) = P|\mu_k| \phi\left(x, \frac{z}{P|\mu_k|}\right), \quad (x, z) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Then

$$\nabla_z \varphi(x, z)^\top = \nabla_z \phi\left(x, \frac{z}{P|\mu_k|}\right)^\top,$$

which leads to

$$\nabla_z \phi(s, z)^\top = \nabla_z \varphi(x, P|\mu_k|z)^\top.$$

Using $\varphi(\cdot)$, equation (4.12) can be written as

$$(4.13) \quad \nabla_z^\top \left[\left(\chi_{[0,\alpha]}(\{\langle P|\mu_k|z, \tilde{\mu} \rangle\}) B(x) + \chi_{[\alpha,1]}(\{\langle P|\mu_k|z, \tilde{\mu} \rangle\}) C(x) \right) \left(I + \nabla_z \varphi(x, P|\mu_k|z)^\top \right) \right] = 0,$$

which is equivalent to the following:

$$(4.14) \quad \nabla_z^\top \left[\left(\chi_{[0,\alpha]}(\{\langle z, \tilde{\mu} \rangle\}) B(x) + \chi_{[\alpha,1]}(\{\langle z, \tilde{\mu} \rangle\}) C(x) \right) \left(I + \nabla_z \varphi(x, z)^\top \right) \right] = 0.$$

Since $\tilde{\mu}$ is a unit vector, there exists an orthogonal matrix Q such that $\tilde{\mu} = Q^\top e_1$ with $e_1 = (1, 0, \dots, 0)^\top \in \mathbb{R}^n$. Let $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)^\top = Qz$, and $\tilde{\varphi}(x, \tilde{z}) = \varphi(x, Q^\top \tilde{z}) \equiv \varphi(x, z)$. Then $\tilde{z}_1 = \langle \tilde{z}, e_1 \rangle = \langle Qz, Q\tilde{\mu} \rangle = \langle z, \tilde{\mu} \rangle$, and

$$\nabla_z \varphi(x, z)^\top = Q^\top \nabla_{\tilde{z}} \varphi(x, Q^\top \tilde{z})^\top = Q^\top \nabla_{\tilde{z}} \tilde{\varphi}(x, \tilde{z})^\top.$$

Thus,

$$(4.15) \quad \nabla_{\tilde{z}}^\top \left[Q \left(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x) \right) \left(I + Q^\top \nabla_{\tilde{z}} \tilde{\varphi}(x, \tilde{z})^\top \right) \right] = 0.$$

Since $Q(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x))$, the coefficient of the above equation, is independent of $\tilde{z}_2, \tilde{z}_3, \dots, \tilde{z}_n$, by the uniqueness, the solution $\tilde{\varphi}(x, \cdot)$ of equation (4.15) must be independent of $\tilde{z}_2, \tilde{z}_3, \dots, \tilde{z}_n$. Thus (4.15) further implies

$$(4.16) \quad \frac{\partial}{\partial \tilde{z}_1} \left[e_1^\top Q \left(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x) \right) \left(I + \tilde{\mu} \left(\frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1) \right)^\top \right) \right] = 0.$$

Hence, there exists a constant vector $X \in \mathbb{R}^n$ such that (note $Q^\top e_1 = \tilde{\mu}$)

$$\begin{aligned} X^\top &= e_1^\top Q \left(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x) \right) \left[I + \tilde{\mu} \left(\frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1) \right)^\top \right] \\ &= \tilde{\mu}^\top \left(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x) \right) \\ &\quad + \tilde{\mu}^\top \left(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x) \right) \tilde{\mu} \left(\frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1) \right)^\top. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1) &= \frac{X - \left(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x) \right) \tilde{\mu}}{\tilde{\mu}^\top \left(\chi_{[0,\alpha]}(\{\tilde{z}_1\}) B(x) + \chi_{[\alpha,1]}(\{\tilde{z}_1\}) C(x) \right) \tilde{\mu}} \\ &= \frac{X - B(x) \tilde{\mu}}{\tilde{\mu}^\top B(x) \tilde{\mu}} \chi_{[0,\alpha]}(\{\tilde{z}_1\}) + \frac{X - C(x) \tilde{\mu}}{\tilde{\mu}^\top C(x) \tilde{\mu}} \chi_{[\alpha,1]}(\{\tilde{z}_1\}). \end{aligned}$$

Since $\phi(x, \cdot) \in W_{\#}^{1,2}((0, 1)^n; \mathbb{R}^n)/\mathbb{R}^n$, for $\tilde{z} = (P^2|\mu_k|^2, \tilde{z}_2, \dots, \tilde{z}_n)^\top$, we have

$$\begin{aligned}\tilde{\varphi}(x, \tilde{z}) &= \tilde{\varphi}(x, P^2|\mu_k|^2 e_1) = \varphi(x, P^2|\mu_k|^2 Q^\top e_1) = \varphi(x, P^2|\mu_k|^2 \tilde{\mu}) \\ &= P|\mu_k|\phi(x, P\mu_k) = P|\mu_k|\phi(x, 0) = \varphi(x, 0) = \tilde{\varphi}(x, 0).\end{aligned}$$

Hence,

$$\begin{aligned}0 &= \frac{\tilde{\varphi}(x, P^2|\mu_k|^2 e_1) - \tilde{\varphi}(x, 0)}{P^2|\mu_k|^2} \\ &= \frac{1}{P^2|\mu_k|^2} \int_0^{P^2|\mu_k|^2} \left[\frac{X - B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} \chi_{[0, \alpha)}(\{\tilde{z}_1\}) + \frac{X - C(x)\tilde{\mu}}{\tilde{\mu}^\top C(x)\tilde{\mu}} \chi_{[\alpha, 1)}(\{\tilde{z}_1\}) \right] d\tilde{z}_1 \\ &= \left(\frac{\alpha}{\tilde{\mu}^\top B(x)\tilde{\mu}} + \frac{1 - \alpha}{\tilde{\mu}^\top C(x)\tilde{\mu}} \right) X - \left(\frac{\alpha B(x)}{\tilde{\mu}^\top B(x)\tilde{\mu}} + \frac{(1 - \alpha)C(x)}{\tilde{\mu}^\top C(x)\tilde{\mu}} \right) \tilde{\mu}.\end{aligned}$$

This yields

$$X = \frac{\alpha[\tilde{\mu}^\top C(x)\tilde{\mu}]B(x)\tilde{\mu} + (1 - \alpha)[\tilde{\mu}^\top B(x)\tilde{\mu}]C(x)\tilde{\mu}}{\tilde{\mu}^\top [\alpha C(x) + (1 - \alpha)B(x)]\tilde{\mu}},$$

and for any $\tilde{z} \in \mathbb{R}^n$ with $\tilde{z}_1 \in [0, \alpha)$,

$$\begin{aligned}\tilde{\varphi}(x, \tilde{z}) &= \tilde{\varphi}(x, \tilde{z}_1 e_1) = \tilde{\varphi}(x, 0) + \int_0^{\tilde{z}_1} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tau e_1) d\tau \\ &= \tilde{\varphi}(x, 0) + \int_0^{\tilde{z}_1} \frac{X - B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} d\tau = \tilde{\varphi}(x, 0) + \tilde{z}_1 \frac{X - B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}}.\end{aligned}$$

For $\tilde{z}_1 \in [\alpha, 1)$, we have

$$\begin{aligned}\tilde{\varphi}(x, \tilde{z}) &= \tilde{\varphi}(x, \alpha e_1) = \tilde{\varphi}(x, 0) + \alpha \frac{X - B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} + \int_\alpha^{\tilde{z}_1} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tau e_1) d\tau \\ &= \tilde{\varphi}(x, 0) + \alpha \frac{X - B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} + \int_0^{\tilde{z}_1} \frac{X - C(x)\tilde{\mu}}{\tilde{\mu}^\top C(x)\tilde{\mu}} d\tau \\ &= \tilde{\varphi}(x, 0) + \alpha \frac{X - B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} + (\tilde{z}_1 - \alpha) \frac{X - C(x)\tilde{\mu}}{\tilde{\mu}^\top C(x)\tilde{\mu}}.\end{aligned}$$

As a result,

$$\begin{aligned}\tilde{\varphi}(x, e_1) - \tilde{\varphi}(x, 0) &= \alpha \frac{X - B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} + (1 - \alpha) \frac{X - C(x)\tilde{\mu}}{\tilde{\mu}^\top C(x)\tilde{\mu}} \\ &= \left(\frac{\alpha}{\tilde{\mu}^\top B(x)\tilde{\mu}} + \frac{(1 - \alpha)}{\tilde{\mu}^\top C(x)\tilde{\mu}} \right) X - \left[\frac{\alpha B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} + \frac{(1 - \alpha)C(x)\tilde{\mu}}{\tilde{\mu}^\top C(x)\tilde{\mu}} \right] \\ &= \frac{\alpha \tilde{\mu}^\top C(x)\tilde{\mu} + (1 - \alpha) \tilde{\mu}^\top B(x)\tilde{\mu}}{[\tilde{\mu}^\top B(x)\tilde{\mu}][\tilde{\mu}^\top C(x)\tilde{\mu}]} \frac{\alpha[\tilde{\mu}^\top C(x)\tilde{\mu}]B(x)\tilde{\mu} + (1 - \alpha)[\tilde{\mu}^\top B(x)\tilde{\mu}]C(x)\tilde{\mu}}{\tilde{\mu}^\top [\alpha C(x) + (1 - \alpha)B(x)]\tilde{\mu}} \\ &\quad - \left[\frac{\alpha B(x)\tilde{\mu}}{\tilde{\mu}^\top B(x)\tilde{\mu}} + \frac{(1 - \alpha)C(x)\tilde{\mu}}{\tilde{\mu}^\top C(x)\tilde{\mu}} \right] = 0.\end{aligned}$$

Thus, $\tilde{z}_1 \mapsto \tilde{\varphi}(x, \tilde{z}_1 e_1)$ is 1-periodic. On the other hand, for any $z \in [0, 1]^n$,

$$\begin{aligned}\phi(x, z) &= \frac{1}{P|\mu_k|} \varphi(x, P|\mu_k|z) = \frac{1}{P|\mu_k|} \tilde{\varphi}(x, P|\mu_k|Qz) \\ &= \frac{1}{P|\mu_k|} \tilde{\varphi}(x, P|\mu_k|(e_1^\top Qz)e_1) = \frac{1}{P|\mu_k|} \tilde{\varphi}(x, P|\mu_k|(\tilde{\mu}^\top z)e_1) = \frac{1}{P|\mu_k|} \tilde{\varphi}(x, P|\mu_k|(e_1 \tilde{\mu}^\top)z).\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla_z \phi(x, z)^\top &= \tilde{\mu} e_1^\top \nabla_{\tilde{z}} \tilde{\varphi}(x, (\tilde{\mu}^\top z) P | \mu_k | e_1)^\top = \tilde{\mu} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, P(\mu_k^\top z) e_1)^\top = \tilde{\mu} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \{\langle P \mu_k, z \rangle\} e_1)^\top \\
&= \tilde{\mu} \frac{X^\top - \tilde{\mu}^\top \left(\chi_{[0, \alpha]}(\{\langle P \mu_k, z \rangle\}) B(x) + \chi_{[\alpha, 1]}(\{\langle P \mu_k, z \rangle\}) C(x) \right)}{\tilde{\mu}^\top \left(\chi_{[0, \alpha]}(\{\langle P \mu_k, z \rangle\}) B(x) + \chi_{[\alpha, 1]}(\{\langle P \mu_k, z \rangle\}) C(x) \right) \tilde{\mu}} \\
&= \frac{\tilde{\mu} [X^\top - \tilde{\mu}^\top B(x)]}{\tilde{\mu}^\top B(x) \tilde{\mu}} \chi_{[0, \alpha]}(\{\langle P \mu_k, z \rangle\}) + \frac{\tilde{\mu} [X^\top - \tilde{\mu}^\top C(x)]}{\tilde{\mu}^\top C(x) \tilde{\mu}} \chi_{[\alpha, 1]}(\{\langle P \mu_k, z \rangle\}).
\end{aligned}$$

Therefore, making use of Lemma 4.2, one has

$$\begin{aligned}
\hat{G}(x) &= \int_{[0, 1]^n} G(x, Pz) \left[I + \nabla_z \phi(x, z)^\top \right] dz \\
&= \int_{[0, 1]^n} \left[B(x) \chi_{[0, \alpha]}(\{\langle P \mu_k, z \rangle\}) + C(x) \chi_{[\alpha, 1]}(\{\langle P \mu_k, z \rangle\}) \right] \\
&\quad \cdot \left[I + \frac{\tilde{\mu} [X^\top - \tilde{\mu}^\top B(x)]}{\tilde{\mu}^\top B(x) \tilde{\mu}} \chi_{[0, \alpha]}(\{\langle P \mu_k, z \rangle\}) + \frac{\tilde{\mu} [X^\top - \tilde{\mu}^\top C(x)]}{\tilde{\mu}^\top C(x) \tilde{\mu}} \chi_{[\alpha, 1]}(\{\langle P \mu_k, z \rangle\}) \right] dz \\
&= \int_{[0, 1]^n} \left[\left(B(x) + \frac{B(x) \tilde{\mu} [X^\top - \tilde{\mu}^\top B(x)]}{\tilde{\mu}^\top B(x) \tilde{\mu}} \right) \chi_{[0, \alpha]}(\{\langle P \mu_k, z \rangle\}) \right. \\
&\quad \left. + \left(C(x) + \frac{C(x) \tilde{\mu} [X^\top - \tilde{\mu}^\top C(x)]}{\tilde{\mu}^\top C(x) \tilde{\mu}} \right) \chi_{[\alpha, 1]}(\{\langle P \mu_k, z \rangle\}) \right] dz \\
&= \alpha B(x) + (1 - \alpha) C(x) + \alpha \frac{B(x) \tilde{\mu} [X^\top - \tilde{\mu}^\top B(x)]}{\tilde{\mu}^\top B(x) \tilde{\mu}} + (1 - \alpha) \frac{C(x) \tilde{\mu} [X^\top - \tilde{\mu}^\top C(x)]}{\tilde{\mu}^\top C(x) \tilde{\mu}}.
\end{aligned}$$

Now, we simplify the expression of $\hat{G}(x)$, suppressing x ,

$$\begin{aligned}
B \tilde{\mu} [X^\top - \tilde{\mu}^\top B] &= B \tilde{\mu} \frac{\alpha [\tilde{\mu}^\top C \tilde{\mu}] \tilde{\mu}^\top B + (1 - \alpha) [\tilde{\mu}^\top B \tilde{\mu}] \tilde{\mu}^\top C}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}} - B \tilde{\mu} \tilde{\mu}^\top B \\
&= \frac{\alpha [\tilde{\mu}^\top C \tilde{\mu}] B \tilde{\mu} \tilde{\mu}^\top B + (1 - \alpha) [\tilde{\mu}^\top B \tilde{\mu}] B \tilde{\mu} \tilde{\mu}^\top C - \alpha [\tilde{\mu}^\top C \tilde{\mu}] B \tilde{\mu} \tilde{\mu}^\top B - (1 - \alpha) [\tilde{\mu}^\top B \tilde{\mu}] B \tilde{\mu} \tilde{\mu}^\top B}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}} \\
&= \frac{(1 - \alpha) [\tilde{\mu}^\top B \tilde{\mu}] B \tilde{\mu} \tilde{\mu}^\top (C - B)}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
C \tilde{\mu} [X^\top - \tilde{\mu}^\top C] &= C \tilde{\mu} \frac{\alpha [\tilde{\mu}^\top C \tilde{\mu}] \tilde{\mu}^\top B + (1 - \alpha) [\tilde{\mu}^\top B \tilde{\mu}] \tilde{\mu}^\top C}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}} - C \tilde{\mu} \tilde{\mu}^\top C \\
&= \frac{\alpha [\tilde{\mu}^\top C \tilde{\mu}] C \tilde{\mu} \tilde{\mu}^\top B + (1 - \alpha) [\tilde{\mu}^\top B \tilde{\mu}] C \tilde{\mu} \tilde{\mu}^\top C - \alpha [\tilde{\mu}^\top C \tilde{\mu}] C \tilde{\mu} \tilde{\mu}^\top C - (1 - \alpha) [\tilde{\mu}^\top B \tilde{\mu}] C \tilde{\mu} \tilde{\mu}^\top C}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}} \\
&= \frac{\alpha [\tilde{\mu}^\top C \tilde{\mu}] C \tilde{\mu} \tilde{\mu}^\top (B - C)}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\hat{G} &= \alpha B + (1 - \alpha) C + \alpha \frac{(1 - \alpha) B \tilde{\mu} \tilde{\mu}^\top (C - B)}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}} + (1 - \alpha) \frac{\alpha C \tilde{\mu} \tilde{\mu}^\top (B - C)}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}} \\
&= \alpha B + (1 - \alpha) C - \alpha(1 - \alpha) \frac{(B - C) \tilde{\mu} \tilde{\mu}^\top (C - B)}{\tilde{\mu}^\top [\alpha C + (1 - \alpha) B] \tilde{\mu}}.
\end{aligned}$$

This means that for any $x \in E_k$,

$$\begin{aligned}
(4.17) \quad G(x) &= \alpha B(x) + (1 - \alpha) C(x) - \alpha(1 - \alpha) \frac{(C(x) - B(x)) \mu_k \mu_k^\top (C(x) - B(x))}{\mu_k^\top (\alpha C(x) + (1 - \alpha) B(x)) \mu_k} \\
&= \alpha B(x) + (1 - \alpha) C(x) - \alpha(1 - \alpha) \frac{(C(x) - B(x)) \ell(x) \ell(x)^\top (C(x) - B(x))}{\ell(x)^\top (\alpha C(x) + (1 - \alpha) B(x)) \ell(x)}.
\end{aligned}$$

The proof is completed. \square

By Lemma 4.1, we know that the function $G : \Omega \rightarrow \mathbb{S}_+^n$ appears in the above lemma satisfies the following:

$$\lambda|\xi|^2 \leq \langle G(x)\xi, \xi \rangle \leq \Lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in S^{n-1}.$$

Lemma 4.4. *Let (S1)–(S5) hold and $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C). Let $u(\cdot) \in \mathcal{U}$, $\mu_1, \mu_2, \dots, \mu_m \in \mathbb{R}^n \setminus \{0\}$ and E_1, E_2, \dots, E_m be mutually disjoint measurable sets such that $\bigcup_{i=1}^m E_i = \Omega$. Define*

$$\ell(\cdot) = \sum_{k=1}^m \mu_k \chi_{E_k}(\cdot).$$

Let $\alpha \in (0, 1)$ and $y^\alpha(\cdot; \ell(\cdot))$ be the weak solution of

$$(4.18) \quad \begin{cases} -\nabla \cdot (\widehat{G}(x) \nabla y^\alpha(x; \ell(\cdot))) = \alpha f(x, y^\alpha(x; \ell(\cdot)), u(x)) + (1 - \alpha) f(x, y^\alpha(x; \ell(\cdot)), u(x)), & \text{in } \Omega, \\ y^\alpha(x; \ell(\cdot)) = 0, & x \in \partial\Omega, \end{cases}$$

with $\widehat{G}(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n)$ being given by

$$(4.19) \quad \begin{aligned} \widehat{G}(x) &= \alpha A(x, u(x)) + (1 - \alpha) A(x, \bar{u}(x)) \\ &\quad - \frac{\alpha(1 - \alpha) [A(x, u(x)) - A(x, \bar{u}(x))] \ell(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{(1 - \alpha) \ell(x)^\top A(x, u(x)) \ell(x) + \alpha \ell(x)^\top A(x, \bar{u}(x)) \ell(x)}, \quad \forall x \in \Omega. \end{aligned}$$

Then

$$(4.20) \quad J^\alpha(u(\cdot), \ell(\cdot)) \equiv \int_{\Omega} \left(\alpha f^0(x, y^\alpha(x; \ell(\cdot)), u(x)) + (1 - \alpha) f^0(x, y^\alpha(x; \ell(\cdot)), u(x)) \right) dx \geq J(\bar{u}(\cdot)).$$

Proof. We split the proof into two steps.

Step I. First, let all the components of $\mu_1, \mu_2, \dots, \mu_m \in \mathbb{Q}^n \setminus \{0\}$. Let P be a positive integer such that all the components of $P\mu_1, P\mu_2, \dots, P\mu_m$ are integers. For $\varepsilon > 0$, define

$$(4.21) \quad u^{\alpha, \varepsilon}(x) = \begin{cases} u(x), & \text{if } \left\{ \left\langle \frac{x}{\varepsilon}, \mu_k \right\rangle \right\} \in [0, \alpha), \quad x \in E_k, \\ \bar{u}(x), & \text{if } \left\{ \left\langle \frac{x}{\varepsilon}, \mu_k \right\rangle \right\} \in [\alpha, 1), \quad x \in E_k. \end{cases}$$

Let $y^{\alpha, \varepsilon}(\cdot) = y(\cdot; u^{\alpha, \varepsilon}(\cdot))$ be the solution to the state equation (1.1) corresponding to the control $u^{\alpha, \varepsilon}(\cdot)$. It is standard that $y^{\alpha, \varepsilon}(\cdot)$ is uniformly bounded in $H_0^1(\Omega)$ and $L^\infty(\Omega)$. Thus, along a subsequence $\varepsilon \downarrow 0$, $y^{\alpha, \varepsilon}(\cdot)$ converges to some $z^\alpha(\cdot)$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Let $\tilde{y}^{\alpha, \varepsilon}(\cdot) \in H_0^1(\Omega)$ be the weak solution of

$$(4.22) \quad \begin{cases} -\nabla \cdot (A(x, u^{\alpha, \varepsilon}(x)) \nabla \tilde{y}^{\alpha, \varepsilon}(x)) = \alpha f(x, z^\alpha(x), u(x)) + (1 - \alpha) f(x, z^\alpha(x), u(x)), & \text{in } \Omega, \\ \tilde{y}^{\alpha, \varepsilon}(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

Then $\hat{y}^{\alpha, \varepsilon}(\cdot) \equiv y^{\alpha, \varepsilon}(\cdot) - \tilde{y}^{\alpha, \varepsilon}(\cdot)$ satisfies

$$(4.23) \quad \begin{cases} -\nabla \cdot (A(x, u^{\alpha, \varepsilon}(x)) \nabla \hat{y}^{\alpha, \varepsilon}(x)) = f(x, y^{\alpha, \varepsilon}(x), u^{\alpha, \varepsilon}(x)) \\ \quad - \alpha f(x, z^\alpha(x), u(x)) - (1 - \alpha) f(x, z^\alpha(x), u(x)), & \text{in } \Omega, \\ \hat{y}^{\alpha, \varepsilon}(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

By (S3) and the boundedness of $y^{\alpha, \varepsilon}(\cdot)$ in $L^\infty(\Omega)$, we can see that

$$f(\cdot, y^{\alpha, \varepsilon}(\cdot), u^{\alpha, \varepsilon}(\cdot)) - f(\cdot, z^\alpha(\cdot), u^{\alpha, \varepsilon}(\cdot))$$

converges strongly in $L^2(\Omega)$. In addition, by Riemann-Lebesgue's Theorem, we see that

$$f(\cdot, y^\alpha(\cdot; \ell(\cdot)), u^{\alpha, \varepsilon}(\cdot)) \rightarrow \alpha f(\cdot, z^\alpha(\cdot), u(\cdot)) + (1 - \alpha)f(\cdot, z^\alpha(\cdot), u(\cdot)), \quad \text{weakly in } L^2(\Omega) \text{ as } \varepsilon \rightarrow 0.$$

Then, (S2) and (4.23) imply that $\hat{y}^{\alpha, \varepsilon}(\cdot)$ converges weakly to zero in $H_0^1(\Omega)$ (as $\varepsilon \downarrow 0$). Consequently, along a subsequence $\varepsilon \downarrow 0$, $\hat{y}^{\alpha, \varepsilon}(\cdot)$ converges to $z^\alpha(\cdot)$ weakly in $H_0^1(\Omega)$. Combining this with Lemma 4.3, we get $z^\alpha(\cdot) = y^\alpha(\cdot; \ell(\cdot))$, where $y^\alpha(\cdot; \ell(\cdot))$ is the weak solution of (4.18). This implies $y^{\alpha, \varepsilon}(\cdot)$ itself converges to $y^\alpha(\cdot; \ell(\cdot))$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ (as $\varepsilon \downarrow 0$). Hence,

$$(4.24) \quad \begin{aligned} J(\bar{u}(\cdot)) &\leq \lim_{\varepsilon \downarrow 0} \int_{\Omega} f^0(x, y^{\alpha, \varepsilon}(x), u^{\alpha, \varepsilon}(x)) dx = \lim_{\varepsilon \downarrow 0} \int_{\Omega} f^0(x, y^\alpha(x; \ell(\cdot)), u^{\alpha, \varepsilon}(x)) dx \\ &= \int_{\Omega} \left(\alpha f^0(x, y^\alpha(x; \ell(\cdot)), u(x)) + (1 - \alpha)f^0(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) \right) dx = J^\alpha(u(\cdot), \ell(\cdot)), \end{aligned}$$

proving (4.20) for the case of $\ell(\cdot)$ valued in rational numbers.

Step II. Now, let $\mu_1, \dots, \mu_m \in \mathbb{R}^n \setminus \{0\}$. We can select $\mu_1^\lambda, \mu_2^\lambda, \dots, \mu_m^\lambda \in \mathbb{Q}^n \setminus \{0\}$ such that $\mu_k^\lambda \rightarrow \mu_k$ as $\lambda \rightarrow +\infty$. By Step I, one has

$$(4.25) \quad \int_{\Omega} \left(\alpha f^0(x, y_\lambda^\alpha(x), u(x)) + (1 - \alpha)f^0(x, y_\lambda^\alpha(x), u(x)) \right) dx \geq J(\bar{u}(\cdot)),$$

where

$$(4.26) \quad \begin{cases} -\nabla \cdot (G_\lambda(x) \nabla y_\lambda^\alpha(x)) = \alpha f(x, y_\lambda^\alpha(x), u(x)) + (1 - \alpha)f(x, y_\lambda^\alpha(x), u(x)), & \text{in } \Omega, \\ y_\lambda^\alpha|_{\partial\Omega} = 0 \end{cases}$$

with $G_\lambda(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n)$ given by

$$(4.27) \quad \begin{aligned} G_\lambda(x) &= \alpha A(x, u(x)) + (1 - \alpha)A(x, \bar{u}(x)) \\ &- \frac{\alpha(1 - \alpha)[A(x, u(x)) - A(x, \bar{u}(x))] \mu_k^\lambda (\mu_k^\lambda)^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{(1 - \alpha)(\mu_k^\lambda)^\top A(x, u(x)) \mu_k^\lambda + \alpha(\mu_k^\lambda)^\top A(x, \bar{u}(x)) \mu_k^\lambda}, \quad \forall x \in E_k. \end{aligned}$$

By Lemma 4.1, $G_\lambda(x) \in \mathbb{S}_+^n$ for almost all $x \in \Omega$. Clearly, $G_\lambda(\cdot)$ converges to $G(\cdot)$ strongly in $L^\infty(\Omega)$. Then by Lemma 3.2 and a standard argument, we have the convergence of $y_\lambda^\alpha(\cdot)$ to $y^\alpha(\cdot)$ strongly in $H_0^1(\Omega)$. Then (4.20) follows. \square

We now turn to a proof of Proposition 3.3.

Proof of Proposition 3.3. By Luzin's Theorem, for any integer $\lambda \geq 1$, there exists a closed subset F_λ of $\bar{\Omega}$, such that $\ell(\cdot)$ is continuous on F_λ and $|\bar{\Omega} \setminus F_\lambda| \leq \frac{1}{\lambda}$, where $|S|$ stands for the Lebesgue measure of the set S . Since F_λ is also bounded, $\ell(\cdot)$ is uniformly continuous on F_λ . Thus, there exist disjoint measurable sets $E_{\lambda 1}, E_{\lambda 2}, \dots, E_{\lambda m_\lambda}$ such that

$$\bigcup_{k=1}^{m_\lambda} E_{\lambda k} = F_\lambda, \quad \sup_{x, \tilde{x} \in E_{\lambda k}} |\ell(x) - \ell(\tilde{x})| \leq \frac{1}{\lambda}, \quad \forall j = 1, 2, \dots, m_\lambda.$$

Choosing arbitrary $x^{\lambda k}$ from $E_{\lambda k}$ and $x^{\lambda 0}$ from $E_{\lambda 0} \equiv \bar{\Omega} \setminus F_\lambda$, we define

$$\mu_{\lambda k} = \ell(x^{\lambda k}), \quad k = 0, 1, 2, \dots, m_\lambda.$$

By Lemma 4.3,

$$(4.28) \quad \int_{\Omega} \left(\alpha f^0(x, y_\lambda^\alpha(x), u(x)) + (1 - \alpha)f^0(x, y_\lambda^\alpha(x), u(x)) \right) dx \geq J(\bar{u}(\cdot)),$$

where $y_\lambda^\alpha(\cdot)$ is the weak solution of the following:

$$(4.29) \quad \begin{cases} -\nabla \cdot (G_\lambda^\alpha(x) \nabla y_\lambda^\alpha(x)) = \alpha f(x, y_\lambda^\alpha(x), u(x)) + (1-\alpha) f(x, y_\lambda^\alpha(x), \bar{u}(x)), & \text{in } \Omega, \\ y_\lambda^\alpha|_{\partial\Omega} = 0, \end{cases}$$

with $G_\lambda^\alpha(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n)$ being given by

$$(4.30) \quad \begin{aligned} G_\lambda^\alpha(x) &= \alpha A(x, u(x)) + (1-\alpha) A(x, \bar{u}(x)) \\ &- \frac{\alpha(1-\alpha) [A(x, u(x)) - A(x, \bar{u}(x))] \mu_{\lambda k} \mu_{\lambda k}^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{(1-\alpha) \mu_{\lambda k}^\top A(x, u(x)) \mu_{\lambda k} + \alpha \mu_{\lambda k}^\top A(x, \bar{u}(x)) \mu_{\lambda k}}, \\ &\quad \forall x \in E_{\lambda k}, \quad k = 0, 1, 2, \dots, m_\lambda. \end{aligned}$$

Obviously, $G_\lambda^\alpha(\cdot)$ converges to $A^\alpha(\cdot)$ strongly in $L^2(\Omega)$, as $\lambda \rightarrow \infty$, where $A^\alpha(\cdot)$ is defined by (3.21). Thus it follows from (S2)–(S3) and Lemma 3.2 that $y_\lambda^\alpha(\cdot)$ converges to $y^\alpha(\cdot; \ell(\cdot))$, as $\lambda \rightarrow \infty$, strongly in $H_0^1(\Omega)$. We then obtain (3.24). \square

5 Second-Order Necessary Conditions.

In this section, we are going to prove Theorem 2.4. For readers' convenience, we will rewrite the relevant equations when needed. We first establish the following lemma.

Lemma 5.1. *Let (S1)–(S5) hold. Let $\bar{u}(\cdot), u(\cdot) \in \mathcal{U}$ and $\ell(\cdot) \in \mathcal{L}$. For any $\alpha \in (0, 1)$, define*

$$(5.1) \quad \begin{aligned} A^\alpha(x; \ell(\cdot)) &= \alpha A(x, u(x)) + (1-\alpha) A(x, \bar{u}(x)) \\ &- \frac{\alpha(1-\alpha) [A(x, u(x)) - A(x, \bar{u}(x))] \ell(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{(1-\alpha) \ell(x)^\top A(x, u(x)) \ell(x) + \alpha \ell(x)^\top A(x, \bar{u}(x)) \ell(x)}. \end{aligned}$$

Let $\bar{y}(\cdot), y^\alpha(\cdot; \ell(\cdot))$ be the weak solutions of the following equations:

$$(5.2) \quad \begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla \bar{y}(x)) = f(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ \bar{y}(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$(5.3) \quad \begin{cases} -\nabla \cdot (A^\alpha(x; \ell(\cdot)) \nabla y^\alpha(x; \ell(\cdot))) = (1-\alpha) f(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) + \alpha f(x, y^\alpha(x; \ell(\cdot)), u(x)), & \text{in } \Omega, \\ y^\alpha(x; \ell(\cdot)) = 0, & \text{in } \partial\Omega, \end{cases}$$

respectively. Then, as $\alpha \downarrow 0$, $Y^\alpha \equiv \frac{y^\alpha(\cdot; \ell(\cdot)) - \bar{y}(\cdot)}{\alpha}$ converges to $Y(\cdot)$ weakly in $H_0^1(\Omega)$ with $Y(\cdot)$ being the weak solution of

$$(5.4) \quad \begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla Y(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) Y(x) + \nabla \cdot (\Theta(x) \nabla \bar{y}(x)) \\ \quad + f(x, \bar{y}(x), u(x)) - f(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ Y|_{\partial\Omega} = 0, \end{cases}$$

where

$$(5.5) \quad \begin{aligned} \Theta(x) &= A(x, u(x)) - A(x, \bar{u}(x)) \\ &- \frac{[A(x, u(x)) - A(x, \bar{u}(x))] \ell(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{\ell(x)^\top A(x, u(x)) \ell(x)}. \end{aligned}$$

Proof. Let $y^\alpha(\cdot) = y^\alpha(\cdot; \ell(\cdot))$. We have

$$\begin{aligned}
& -\nabla \cdot (A(x, \bar{u}(x)) \nabla Y^\alpha(x)) \\
&= \frac{f(x, y^\alpha(x), \bar{u}(x)) - f(x, \bar{y}(x), \bar{u}(x))}{\alpha} + f(x, y^\alpha(x), u(x)) - f(x, y^\alpha(x), \bar{u}(x)) \\
&\quad - \nabla \cdot \left(\frac{A(x, \bar{u}(x)) - A(x, u(x))}{\alpha} \nabla y^\alpha(x) \right) \\
(5.6) \quad &= \left(\int_0^1 f_y(x, \bar{y}(x) + t(y^\alpha(x) - \bar{y}(x)), \bar{u}(x)) dt \right) Y^\alpha(x) \\
&\quad + f(x, y^\alpha(x), u(x)) - f(x, y^\alpha(x), \bar{u}(x)) + \nabla \cdot \left((A(x, u(x)) - A(x, \bar{u}(x))) \nabla y^\alpha(x) \right) \\
&\quad - (1 - \alpha) \nabla \cdot \left(\frac{[A(x, u(x)) - A(x, \bar{u}(x))] \ell(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{(1 - \alpha) \ell(x)^\top A(x, u(x)) \ell(x) + \alpha \ell(x)^\top A(x, \bar{u}(x)) \ell(x)} \nabla y^\alpha(x) \right).
\end{aligned}$$

By Proposition 3.3 and (S2)–(S3), one can see that $y^\alpha(\cdot)$ is bounded uniformly in $H_0^1(\Omega)$. Thus, we can prove that $Y^\alpha(\cdot)$ is bounded uniformly in $H_0^1(\Omega)$, and as $\alpha \downarrow 0$, $y^\alpha(\cdot)$ converges to $\bar{y}(\cdot)$ strongly in $H_0^1(\Omega)$, $Y^\alpha(\cdot)$ converges to $Y(\cdot)$ weakly in $H_0^1(\Omega)$ with $Y(\cdot)$ being the weak solution of (5.4). \square

Let us define

$$(5.7) \quad Z^\alpha(\cdot) \equiv \frac{Y^\alpha(\cdot) - Y(\cdot)}{\alpha}.$$

Then it is natural to expect that as $\alpha \downarrow 0$, $Z^\alpha(\cdot) \rightarrow Z(\cdot)$, weakly in $H_0^1(\Omega)$, with $Z(\cdot)$ being the weak solution of the following equation:

$$(5.8) \quad \begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla Z(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) Z(x) + \frac{1}{2} f_{yy}(x, \bar{y}(x), \bar{u}(x)) |Y(x)|^2 \\ \quad + (f_y(x, \bar{y}(x), u(x)) - f_y(x, \bar{y}(x), \bar{u}(x))) Y(x) \\ \quad + \nabla \cdot \left((A(x, u(x)) - A(x, \bar{u}(x))) \nabla Y(x) \right) + \nabla \cdot \left(\Upsilon(x) \nabla \bar{y}(x) \right), & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, \end{cases}$$

where

$$\Upsilon(x) = [A(x, u(x)) - A(x, \bar{u}(x))] \ell(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))] \frac{\ell(x)^\top A(x, \bar{u}(x)) \ell(x)}{[\ell(x)^\top A(x, u(x)) \ell(x)]^2}.$$

However, this seems not to be true when n is a large. The main reason is that $A(\cdot, \bar{u}(\cdot))$ is only bounded and measurable so that we have $W^{1,p}(\Omega)$ estimates of $Z^\alpha(\cdot)$ only for p near 2. Thus, we do not have a weak maximum principle for $Y(\cdot)$, i.e., we usually cannot get uniform boundedness for $Y(\cdot)$ unless $n \leq 2$. Thus, generally, $Y(\cdot)^2$ times an L^∞ function might not be in $H^{-1}(\Omega)$ unless $n \leq 6$. Hence, $Z(\cdot)$ is probably not well-defined for large n . Fortunately, we have the following relation, which will be sufficient for the proof of

our main result—Theorem 2.4. Recalling the definition of $Z^\alpha(\cdot)$, we have

$$\begin{aligned}
& -\nabla \cdot (A(\bar{u}) \nabla Z^\alpha) \\
&= \frac{1}{\alpha} \left[\left(\int_0^1 f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) dt \right) Y^\alpha - f_y(\bar{y}, \bar{u}) Y \right] + \frac{f(y^\alpha, u) - f(\bar{y}, u)}{\alpha} - \frac{f(y^\alpha, \bar{u}) - f(\bar{y}, \bar{u})}{\alpha} \\
& \quad + \frac{1}{\alpha} \left[\nabla \cdot \left((A(u) - A(\bar{u}) \nabla y^\alpha) - (1 - \alpha) \nabla \cdot \left(\frac{[A(u) - A(\bar{u})] \ell \ell^\top [A(u) - A(\bar{u})]}{(1 - \alpha) \ell^\top A(u) \ell + \alpha \ell^\top A(\bar{u}) \ell} \nabla y^\alpha \right) - \nabla \cdot (\Theta \nabla \bar{y}) \right] \right. \\
&= f_y(\bar{y}, \bar{u}) Z^\alpha + \left(\int_0^1 \frac{f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) - f_y(\bar{y}, \bar{u})}{\alpha} dt \right) Y^\alpha \\
& \quad + \frac{f(y^\alpha, u) - f(\bar{y}, u)}{\alpha} - \frac{f(y^\alpha, \bar{u}) - f(\bar{y}, \bar{u})}{\alpha} + \nabla \cdot \left((A(u) - A(\bar{u})) \nabla Y^\alpha \right) \\
(5.9) \quad & \quad + \nabla \cdot \left(\frac{[A(u) - A(\bar{u})] \ell \ell^\top [A(u) - A(\bar{u})]}{(1 - \alpha) \ell^\top A(u) \ell + \alpha \ell^\top A(\bar{u}) \ell} \nabla y^\alpha \right) \\
& \quad - \nabla \cdot \left(\frac{\ell^\top [A(u) - A(\bar{u})] \ell}{\ell^\top A(u) \ell [(1 - \alpha) \ell^\top A(u) \ell + \alpha \ell^\top A(\bar{u}) \ell]} [A(u) - A(\bar{u})] \ell \ell^\top [A(u) - A(\bar{u})] \nabla y^\alpha \right) \\
&= f_y(\bar{y}, \bar{u}) Z^\alpha + \left(\int_0^1 dt \int_0^1 \tau f_{yy}(\bar{y} + \tau t(y^\alpha - \bar{y}), \bar{u}) d\tau \right) |Y^\alpha|^2 \\
& \quad + \left[\int_0^1 \left(f_y(\bar{y} + t(y^\alpha - \bar{y}), u) - f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) \right) dt \right] Y^\alpha + \nabla \cdot \left((A(u) - A(\bar{u})) \nabla Y^\alpha \right) \\
& \quad + \nabla \cdot \left(\frac{\ell^\top A(\bar{u}) \ell}{\ell^\top A(u) \ell [(1 - \alpha) \ell^\top A(u) \ell + \alpha \ell^\top A(\bar{u}) \ell]} [A(u) - A(\bar{u})] \ell \ell^\top [A(u) - A(\bar{u})] \nabla y^\alpha \right).
\end{aligned}$$

Now, we are ready to prove our main result.

Proof of Theorem 2.4. Let $(u(\cdot), \ell(\cdot)) \in \mathcal{V}_0(\bar{u}(\cdot))$ and $\alpha \in (0, 1)$. Let $y^\alpha(\cdot)$ be the weak solution to (3.20), with $A^\alpha(\cdot)$ and $J^\alpha(u(\cdot), \ell(\cdot))$ be defined by (3.21) and (3.22), respectively. Let $Y^\alpha(\cdot) = \frac{y^\alpha(\cdot) - \bar{y}(\cdot)}{\alpha}$. Then $Y^\alpha(\cdot)$ satisfies (5.6). We have

$$\begin{aligned}
& \int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) + f_y^0(\bar{y}, \bar{u}) Y \right) dx \\
&= \int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) \right) dx + \int_\Omega \left(f_y(\bar{y}, \bar{u}) \bar{\psi} + \nabla \cdot (A(\bar{u}) \nabla \bar{\psi}) Y \right) dx \\
(5.10) \quad &= \int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) + f_y(\bar{y}, \bar{u}) Y + \nabla \cdot (A(\bar{u}) \nabla Y) \right) \bar{\psi} dx \\
&= \int_\Omega \left[f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) - \left(f(\bar{y}, u) - f(\bar{y}, \bar{u}) + \nabla \cdot (\Theta \nabla \bar{y}) \right) \bar{\psi} \right] dx \\
&= \int_\Omega \left\{ H(\bar{y}, \bar{\psi}, \nabla \bar{y}, \nabla \bar{\psi}, \bar{u}) - H(\bar{y}, \bar{\psi}, \nabla \bar{y}, \nabla \bar{\psi}, u) - \frac{\langle [A(u) - A(\bar{u})] \nabla \bar{y}, \ell \rangle \langle [A(u) - A(\bar{u})] \nabla \bar{\psi}, \ell \rangle}{\langle A(u) \ell, \ell \rangle} \right\} dx = 0,
\end{aligned}$$

where $\Theta(\cdot)$ is given by (5.5). In fact,

$$\int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) + f_y^0(\bar{y}, \bar{u}) Y(x) \right) dx = \lim_{\alpha \downarrow 0} \frac{J^\alpha(u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot))}{\alpha} = 0,$$

due to the partial singularity of $\bar{u}(\cdot)$ at $(u(\cdot), \ell(\cdot)) \in \mathcal{V}_0(\bar{u}(\cdot))$. This leads to

$$(5.11) \quad \int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) \right) dx = - \int_\Omega f_y^0(\bar{y}, \bar{u}) Y dx.$$

Then, using (5.10), (2.6) and (5.6), we have (suppressing x whenever no confusion would be caused, for

notational simplicity)

$$\begin{aligned}
& J^\alpha(u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot)) \equiv \alpha \int_{\Omega} f^0(y^\alpha, u) dx + (1 - \alpha) \int_{\Omega} f^0(y^\alpha, \bar{u}) dx - \int_{\Omega} f^0(\bar{y}, \bar{u}) dx \\
& = \alpha \int_{\Omega} \left(f^0(y^\alpha, u) - f^0(y^\alpha, \bar{u}) \right) dx + \int_{\Omega} \left(f^0(y^\alpha, \bar{u}) - f^0(\bar{y}, \bar{u}) \right) dx \\
& = \alpha \int_{\Omega} \left(f^0(y^\alpha, u) - f^0(\bar{y}, u) - f^0(y^\alpha, \bar{u}) + f^0(\bar{y}, \bar{u}) + f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) \right) dx \\
& \quad + \int_{\Omega} \left(f^0(y^\alpha, \bar{u}) - f^0(\bar{y}, \bar{u}) \right) dx \\
& = \alpha^2 \int_{\Omega} \left[\int_0^1 \left(f_y^0(\bar{y} + \alpha t Y^\alpha, u) - f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) \right) dt \right] Y^\alpha dx \\
& \quad + \alpha \int_{\Omega} \left[\left(\int_0^1 f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) dt \right) Y^\alpha - f_y^0(\bar{y}, \bar{u}) Y \right] dx \\
& = \alpha^2 \int_{\Omega} \left[\int_0^1 \left(f_y^0(\bar{y} + \alpha t Y^\alpha, u) - f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) \right) dt \right] Y^\alpha dx \\
& \quad + \alpha \int_{\Omega} \left[\left(\int_0^1 f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) dt - f_y^0(\bar{y}, \bar{u}) \right) Y^\alpha + \alpha f_y^0(\bar{y}, \bar{u}) Z^\alpha \right] dx \\
& = \alpha^2 \int_{\Omega} \left[\int_0^1 \left(f_y^0(\bar{y} + \alpha t Y^\alpha, u) - f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) \right) dt \right] Y^\alpha dx \\
& \quad + \alpha^2 \int_{\Omega} \left[\int_0^1 \left(\int_0^1 t f_{yy}^0(\bar{y} + \alpha t \tau Y^\alpha, \bar{u}) d\tau \right) dt \right] |Y^\alpha|^2 dx + \alpha^2 \int_{\Omega} \left(f_y(\bar{y}, \bar{u}) \bar{\psi} + \nabla \cdot (A(\bar{u}) \nabla \bar{\psi}) \right) Z^\alpha dx \\
(5.12) \quad & = \alpha^2 \int_{\Omega} \left[\int_0^1 \left(f_y^0(\bar{y} + \alpha t Y^\alpha, u) - f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) \right) dt \right] Y^\alpha dx \\
& \quad + \alpha^2 \int_{\Omega} \left[\int_0^1 \left(\int_0^1 t f_{yy}^0(\bar{y} + \alpha t \tau Y^\alpha, \bar{u}) d\tau \right) dt \right] |Y^\alpha|^2 dx + \alpha^2 \int_{\Omega} \left(f_y(\bar{y}, \bar{u}) Z^\alpha + \nabla \cdot (A(\bar{u}) \nabla Z^\alpha) \right) \bar{\psi} dx \\
& = \alpha^2 \left\{ \int_{\Omega} \left[\int_0^1 \left(f_y^0(\bar{y} + \alpha t Y^\alpha, u) - f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) \right) dt \right] Y^\alpha dx \right. \\
& \quad + \int_{\Omega} \left[\int_0^1 \left(\int_0^1 t f_{yy}^0(\bar{y} + \alpha t \tau Y^\alpha, \bar{u}) d\tau \right) dt \right] |Y^\alpha|^2 dx \\
& \quad - \int_{\Omega} \left[\int_0^1 \left(\int_0^1 t f_{yy}^0(\bar{y} + \alpha t \tau Y^\alpha, \bar{u}) d\tau \right) dt \right] \bar{\psi} |Y^\alpha|^2 dx \\
& \quad - \int_{\Omega} \left[\int_0^1 \left(f_y(\bar{y} + \alpha t Y^\alpha, u) - f_y(\bar{y} + \alpha t Y^\alpha, \bar{u}) \right) dt \right] \bar{\psi} Y^\alpha dx + \int_{\Omega} \langle (A(u) - A(\bar{u})) \nabla Y^\alpha, \nabla \bar{\psi} \rangle dx \\
& \quad \left. + \int_{\Omega} \left\langle \frac{\ell^\top A(\bar{u}) \ell [A(u) - A(\bar{u})] \ell \ell^\top [A(u) - A(\bar{u})] \nabla y^\alpha}{\ell^\top A(u) \ell [(1 - \alpha) \ell^\top A(u) \ell + \alpha \ell^\top A(\bar{u}) \ell]}, \nabla \bar{\psi} \right\rangle dx \right\} \\
& = \alpha^2 \left\{ \int_{\Omega} \left[\int_0^1 \left(H_y(\bar{y} + \alpha t Y^\alpha, \bar{\psi}, \nabla \bar{y}, \nabla \bar{\psi}, \bar{u}) - H_y(\bar{y} + \alpha t Y^\alpha, \bar{\psi}, \nabla \bar{y}, \nabla \bar{\psi}, u) \right) dt \right] Y^\alpha dx \right. \\
& \quad - \int_{\Omega} \left[\int_0^1 \left(\int_0^1 t H_{yy}(\bar{y} + \alpha t \tau Y^\alpha, \bar{\psi}, \nabla \bar{y}, \nabla \bar{\psi}, \bar{u}) d\tau \right) dt \right] |Y^\alpha|^2 dx + \int_{\Omega} \langle (A(u) - A(\bar{u})) \nabla Y^\alpha, \nabla \bar{\psi} \rangle dx \\
& \quad \left. + \int_{\Omega} \left\langle \frac{[\ell^\top A(\bar{u}) \ell] \{ \ell^\top [A(u) - A(\bar{u})] \nabla \bar{\psi} \} \{ \ell^\top [A(u) - A(\bar{u})] \nabla y^\alpha \}}{\ell^\top A(u) \ell [(1 - \alpha) \ell^\top A(u) \ell + \alpha \ell^\top A(\bar{u}) \ell]} dx \right\} \right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\leq \lim_{\alpha \downarrow 0} \frac{J^\alpha(u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot))}{\alpha^2} \\
&= \int_{\Omega} \left[\left(H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) Y(x) \right. \\
&\quad \left. - \frac{1}{2} H_{yy}(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) |Y(x)|^2 + \langle (A(x, u(x)) - A(x, \bar{u}(x))) \nabla \bar{\psi}(x), \nabla Y(x) \rangle \right. \\
&\quad \left. + \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))] \nabla \bar{\psi}(x) \ell(x)^\top [A(x, u(x)) - A(x, \bar{u}(x))] \nabla \bar{y}(x) \frac{[\ell(x)^\top A(x, \bar{u}(x)) \ell(x)]}{[\ell(x)^\top A(x, u(x)) \ell(x)]^2} \right] dx \\
&= \int_{\Omega} \left[\left(H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) Y(x) \right. \\
&\quad \left. - \frac{1}{2} H_{yy}(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) |Y(x)|^2 + \langle (A(x, u(x)) - A(x, \bar{u}(x))) \nabla \bar{\psi}(x), \nabla Y(x) \rangle \right. \\
&\quad \left. + \left(H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) \frac{\ell(x)^\top A(x, \bar{u}(x)) \ell(x)}{\ell(x)^\top A(x, u(x)) \ell(x)} \right] dx.
\end{aligned}$$

This completes the proof. \square

6 Concluding Remarks.

We have established the second-order necessary conditions for the optimal controls of Problem (C). There are some challenging problems left open. We list some of them here, for which we are still working on with our great efforts.

- Construction of suitable examples for which our second-necessary conditions could lead to some optimal solutions.
- The second-order necessary conditions that we obtained looks complicated. Is it possible to have some better forms?
- Extension to fully non-linear equations.

We hope to be able to report some further results before long. Also, any participation of other interested researchers are welcome.

References

- [1] G. Allaire, *Homogenization and two-scale convergence*, *SIAM J. Math. Anal.* 23 (1992), 1482–1518.
- [2] G. Allaire, *Shape Optimization by the Homogenization Method*, Springer, New York, 2002.
- [3] J. F. Bonnans and A. Hermant, *No-gap second-order optimality conditions for optimal control problems with a single state constraint and control*, *Math. Program., Ser. B*, 117 (2009), 21–50.
- [4] J. F. Bonnans and A. Hermant, *Second-order analysis for optimal control problems with pure state constraints and mixed control-state constraints*, *Ann. I. H. Poincaré*, 26 (2009), 561–598.
- [5] E. Casas, *Optimal control in coefficients of elliptic equations with state constraints*, *Appl. Math. Optim.*, 26 (1992), 21–37.
- [6] E. Casas, *Second order analysis for bang-bang control problems of PDEs*, *SIAM J. Control Optim.*, 50 (2012), 2355–2372.

- [7] E. Casas, J. C. de Los Reye and F. Tröltzsch, *Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints*, *SIAM J. Optim.*, 19 (2008), 616–643.
- [8] E. Casas and M. Mateos, *Second order optimality conditions for semilinear elliptic control problems with finitely many state constraints*, *SIAM J. Control Optim.*, 40 (2002), 1431–1454.
- [9] E. Casas and F. Tröltzsch, *Second order necessary optimality conditions for some state-constrained control problems of semilinear elliptic equations*, *Appl. Math. Optim.*, 39 (1999), 211–227.
- [10] E. Casas and F. Tröltzsch, *Second-order necessary and sufficient optimality conditions for optimality conditions for optimization problems and applications to control theory*, *SIAM J. Optim.*, 13 (2002), 406–431.
- [11] E. Casas and F. Tröltzsch, *First- and second-order optimality conditions for a class of optimal control problems with quasilinear elliptic equations*, *SIAM J. Control Optim.*, 48 (2009), 688–718.
- [12] E. Casas and F. Tröltzsch, *Second order analysis for optimal control problems: improving results expected from abstract theory*, *SIAM J. Optim.*, 22 (2012), 261–279.
- [13] E. Casas and F. Tröltzsch, *Second order optimality conditions and their role in PDE control*, *Jahresber Dtsch Math-Ver* (2015), 117:3–44.
- [14] E. Casas, F. Tröltzsch, and A. Unger, *Second-order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations*, *SIAM J. Control Optim.*, 38 (2000), 1369–1391.
- [15] H. O. Fattorini, *Relaxed controls in infinite dimensional systems*, International Series of Numerical Mathematics, 100 (1991), 115–128.
- [16] R. Gamkrelidze, *Principle of Optimal Control Theory*, Plenum Press, New York, 1978.
- [17] R. Gabasov and F. M. Kirillova, *High order necessary conditions for optimality*, *SIAM J. Control*, 10 (1972), 127–168.
- [18] H. J. Kelly, *A second variation test for singular extremals*, *AIAA J.*, 2 (1964), 1380–1382.
- [19] H. W. Knobloch, *Higher Order Necessary Conditions in Optimal Control Theory*, Lecture Notes in Control & Inform. Sci., Springer-Verlag, 1981.
- [20] R. E. Kopp and H. G. Moyer, *Necessary conditions for singular extremals*, *AIAA J.*, 3 (1965), 1439–1444.
- [21] A. J. Krener, *The high order maximal principle and its application to singular extremals*, *SIAM J. Control Optim.*, 15 (1977), 256–293.
- [22] B. Li and H. Lou, *Optimality Conditions for semilinear hyperbolic equations with controls in coefficients*, *Appl. Math. Optim.* 65 (2012), 371–402.
- [23] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, Boston, 1995.
- [24] H. Lou, *Second-order necessary/sufficient optimality conditions for optimal control problems in the absence of linear structure*, *Discrete and Continuous Dynamical Systems-Series B*, 14 (2010), Special Issue, 1445–1464.
- [25] H. Lou, *Optimality conditions for semilinear parabolic equations with controls in leading term*, *ESAIM: Control, Optimization and Calculus of Variations*, 17 (2011), 975–994.

- [26] H. Lou and J. Yong, *Optimality conditions for semilinear elliptic equations with leading term containing controls*, *SIAM J. Control Optim.*, 48 (2009), 2366–2387.
- [27] H. D. Mittelmann, *Verification of second-order sufficient optimality conditions for semilinear elliptic and parabolic control problems*, *Comp. Optim. Appl.*, 20 (2001), 93–110.
- [28] J. P. Raymond and F. Tröltzsch, *Second order sufficient optimality conditions for nonlinear parabolic control problems with state constraints*, *Discrete Contin. Dynam. Systems*, 6 (2000), 431–450.
- [29] A. Rösch and F. Tröltzsch, *Sufficient second-order optimality conditions for a parabolic optimal control problem with pointwise control-state constraints*, *SIAM J. Control Optim.*, 42 (2003), 138–154.
- [30] A. Rösch and F. Tröltzsch, *Sufficient second-order optimality conditions for an elliptic optimal control problem with pointwise control-state constraints*, *SIAM J. Optim.*, 17 (2006), 776–794.
- [31] J. Warga, *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.
- [32] L. Wang and P. He, *Second-order optimality conditions for optimal control problems governed by 3-dimensional Navier-Stokes equations*, *Acta Math. Scientia*, 26B (2006), 729–734.
- [33] A. Zygmund, *Trigonometric Series*, 3rd ed., Cambridge University Press, Cambridge, 2002.